De Sitter space-time, From Snyder model to Yang model

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SanYa December 18, 2011
Outline

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References

1, The Beltrami Model of De Sitter Space

1.1, Why de Sitter Space?

De Sitter space is one of vacuum solutions of Einstein equation with cosmology constant term.

Our universe accelerated expanding and possibly asymptotic to de Sitter space with a tiny positive cosmological constant $\Lambda$.

Precise cosmology challenges the cosmic scale physics based on Einstein General relativity.

There should exist a new kinematics of the cosmic scale physics.

1.2, 4-d de Sitter Spacetime

A de Sitter spacetime with radius $R$ can be regarded as a hyperboloid embedded in a (1+4)-d Mink-space

$$\eta_{AB}\xi^A\xi^B = -R^2 < 0, \ A, B = 0, \ldots, 4.$$  \hspace{1cm} (1)

Where $\eta_{AB} = \text{diag}(+, -,-,-,-)$.

For the $BdS$ model, the Beltrami coordinates in the chart $U_{+4}$ are

$$x^\mu|_{U_{+4}} = R\xi^\mu/\xi^4, \quad \xi^4|_{U_{+4}} > 0,$$  \hspace{1cm} (2)

where $\mu = 0, \ldots, 3$.

where $x^0 = ct$ is the Beltrami-time, and $x^i, i = 1, 2, 3$, are the Beltrami spacial coordinates. In Beltrami de Sitter spacetime we have
\[ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \left[ \frac{\eta_{\mu\nu}}{\sigma(x)} + \frac{\eta_{\mu\lambda} \eta_{\nu\kappa} x^\lambda x^\kappa}{R^2 \sigma^2(x)} \right] dx^\mu dx^\nu, \] (3)

where

\[ \sigma(x) = 1 - R^{-2} x^\nu x_\nu > 0, \quad x_\nu = \eta_{\nu\lambda} x^\lambda. \] (4)

and,

\[ g_{\mu\nu} = \frac{\eta_{\mu\nu}}{\sigma(x)} + \frac{\eta_{\mu\lambda} \eta_{\nu\kappa} x^\lambda x^\kappa}{R^2 \sigma^2(x)} \] (5)

is the Beltrami spacetime metric.
2. De Sitter special relativity

If we consider that the background spacetime of gravity is de Sitter space, then the special relativistic theory should be established in dS spacetime, where we have 2 constants: speed of light $c$ and the radius $R$ of dS space. We use $SR_{cR}$ to denote the special relativity in de Sitter spacetime.

2.1. The Lagrangian-Hamiltonian formulism

We begin with a brief review of the classical mechanics for a free particle in de Sitter special relativity. The Lagrangian is

$$L_{cR} = -m_0c\frac{ds}{dt} = -m_0c\sqrt{g_{\mu\nu}(x)dx^\mu dx^\nu} = -m_0c\sqrt{g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu}, \quad (6)$$

where $\dot{x}^\mu = \frac{d}{dt}x^\mu$, $g_{\mu\nu}(x)$ is Beltrami metric.

Considering the Euler-Lagrangian equation

$$\frac{\partial L_{cR}}{\partial x^i} = \frac{d}{dt} \frac{\partial L_{cR}}{\partial \dot{x}^i}, \quad (7)$$

we obtain the solution of equation of motion for free particle:

$$\ddot{x}^j = 0, \quad \dot{x}^j = \text{constant}. \quad (8)$$

The canonic momenta and the canonic energy (i.e., Hamiltonian) reads

$$\pi_i = \frac{\partial L_{cR}}{\partial \dot{x}^i} = -m_0\sigma(x)\Gamma g_{i\mu}\dot{x}^\mu \quad (9)$$

$$H_{cR} = \sum_{i=1}^{3} \frac{\partial L_{cR}}{\partial \dot{x}^i} \dot{x}^i - L_{cR} = m_0c\sigma(x)\Gamma g_{0\mu}\dot{x}^\mu. \quad (10)$$
where
\[
\Gamma^{-1} \equiv \sigma(x) \frac{ds}{cdt} = \frac{1}{R} \sqrt{(R^2 - \eta_{ij} x^i x^j)(1 + \frac{\eta_{ij} \dot{x}^i \dot{x}^j}{c^2}) + 2t \eta_{ij} x^i \dot{x}^j - \eta_{ij} \dot{x}^i \dot{x}^j t^2 + \frac{(\eta_{ij} x^i \dot{x}^j)^2}{c^2}}. \tag{11}
\]

It is easy to check that
\[
\lim_{R \to \infty} \Gamma = \lim_{x^i \to 0} \Gamma = \gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{12}
\]

And, in the \( R \to \infty \) limit, \( \pi_i \) and \( H_{cR} \) go back to the standard Einstein Special Relativity’s expressions:
\[
\pi_i \big|_{R \to \infty} = \frac{m_0 v_i}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad H_{cR} \big|_{R \to \infty} = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{13}
\]

where \( v_i = -\eta_{ij} \dot{x}^j \).

2.2, Space-time symmetry of de Sitter special relativity and the Neother charges

The space time transformations preserving the Beltrami metric were discovered about 30 years ago by Lu, Zou and Guo (LZG). When we transform from one initial Beltrami frame \( x^\mu \) to another Beltrami frame \( \tilde{x}^\mu \), and when the origin of the new frame is \( a^\mu \) in the original frame, the transformations between them with 10 parameters is as follows

\[
\begin{align*}
\begin{array}{c}
x^\mu \overset{LZG}{\rightarrow} \tilde{x}^\mu = \pm \sigma(a)^{1/2} \sigma(a, x)^{-1} (x^\nu - a^\nu) D^\mu_\nu, \tag{14} \\
D^\mu_\nu = L^\mu_\nu + R^{-2} \eta_{\nu \rho} a^\rho a^\lambda (\sigma(a) + \sigma^{1/2}(a))^{-1} L^\mu_\lambda, \\
L : = (L^\mu_\nu) \in SO(1, 3), \\
\sigma(x) = 1 - \frac{1}{R^2} \eta_{\mu \nu} x^\mu x^\nu,
\end{array}
\end{align*}
\]
\[ \sigma(a, x) = 1 - \frac{1}{R^2} \eta_{\mu\nu} a^\mu x^\nu. \]

It is called as LZG-transformation. Under LZG-transformation, the \( g_{\mu\nu}(x) \) and the action of \( SR_{cR} \) transfer respectively as follows

\[ g_{\mu\nu}(x) \xrightarrow{\text{LZG}} \tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\lambda}{\partial \tilde{x}^\mu} \frac{\partial x^\rho}{\partial \tilde{x}^\nu} g_{\lambda \rho}(x) = g_{\mu\nu}(\tilde{x}), \tag{15} \]

\[ S_{cR} \equiv \int dt L_{cR}(t) = -m_0 c \int dt \frac{\sqrt{g_{\mu\nu}(x)dx^\mu dx^\nu}}{dt} \xrightarrow{\text{LZG}} \tilde{S}_{cR} = S_{cR}. \tag{16} \]

By the mechanics principle, this action invariance indicates that there are 10 conserved Noether charges in \( SR_{cR} \) like the \( SR_c \) case. Those \( SR_{cR} \)-Noether charges are given as follows

- Noether charges for Lorentz boost: \( K_{cR}^i = m_0 \Gamma c (x^i - t \dot{x}^i) \)
- Charges for space – transitions (momenta): \( P_{cR}^i = m_0 \Gamma \dot{x}^i \)
- Charge for time – transition (energy): \( E_{cR} = m_0 c^2 \Gamma \)
- Charges for rotations in space (angular momenta): \( L_{cR}^i = \epsilon_{ijk} x^j p_{cR}^k \)

From

\[ g^{\mu\nu} \pi_\mu \pi_\nu = m_0^2 c^2. \tag{18} \]

we get the dispersion relation in de Sitter space-time:

\[ E_{cR}^2 = m_0^2 c^4 + c^2 p_{cR}^2 + \frac{c^2}{R^2} (L_{cR}^2 - K_{cR}^2), \tag{19} \]

where \( E_{cR}, p_{cR}, L_{cR}, K_{cR} \) are conserved physical energy, momentum, angular-momentum and boost charges in eq.(17) respectively.

When space rotations were neglected temporarily for simplify, the LZG-transformation both due to a Lorentz-like
boost and a space-transition in the $x^1$ direction with parameters $\beta = \dot{x}^1/c$ and $a^1$ respectively and due to a time transition with parameter $a^0$ can be explicitly written as follows: [see: ML Yan et al: Commun.Theor.Phys. 48(2007)27]

$$
t \rightarrow \tilde{t} = \frac{\sqrt{\sigma(a)}}{c \sigma(a,x)} \gamma \left[ ct - \beta x^1 - a^0 + \beta a^1 + \frac{a^0 - \beta a^1 a_0 c t - a^1 x^1 - (a^0)^2 + (a^1)^2}{R^2} \right]_{\sigma(a)+\sqrt{\sigma(a)}}
$$

$$
x^1 \rightarrow \tilde{x}^1 = \frac{\sqrt{\sigma(a)}}{\sigma(a,x)} \gamma \left[ x^1 - \beta c t + \beta a^0 - a^1 + \frac{a^1 - \beta a^0 a_0 c t - a^1 x^1 - (a^0)^2 + (a^1)^2}{R^2} \right]_{\sigma(a)+\sqrt{\sigma(a)}}
$$

$$
x^2 \rightarrow \tilde{x}^2 = \frac{\sqrt{\sigma(a)}}{\sigma(a,x)} x^2
$$

$$
x^3 \rightarrow \tilde{x}^3 = \frac{\sqrt{\sigma(a)}}{\sigma(a,x)} x^3
$$

(20)

It is easy to check when $R \rightarrow \infty$ the above transformation goes back to Poincaré transformation.

And after tedious calculations one gets the velocity transformation formula on $SR_{cR}$-light cone as follows:

$$
u \rightarrow \tilde{u} \equiv \frac{d\tilde{x}^1}{dt} = \frac{ca^0(c - u)A_1 + cR^2 \sigma(ax)}{a^0(c - u)A_0 + R^2 \sigma(ax)} \left[ u - v + \frac{a^0(1-\beta) a^0(c-u)}{R^2} \right]_{\sigma(a)+\sqrt{\sigma(a)}}.
$$

(21)

Therefore, when $u = c$, we have

$$
c \rightarrow \tilde{c} = \frac{c[c - v]}{[c - v]} = c.
$$

(22)

That the light velocity in the $SR_{cR}$-light cone reference systems is an universal constant.
2.3. Is there superluminal particles in 4−d de Sitter spacetime?

From the dispersion relation in de Sitter spacetime, the velocity of a free particle can be written as

\[ v = \frac{\partial E_{cR}}{\partial p_{cR}} = c + \frac{c}{2p_{cR}R^2} \frac{d}{dp_{cR}} [L_{cR}^2 - K_{cR}^2] \sqrt{1 + \frac{m_0^2c^2}{p_{cR}^2} + \frac{1}{R^2p_{cR}^2} [L_{cR}^2 - K_{cR}^2]} \] (23)

Then we have

\[ \frac{v - c}{c} = -\frac{m_0^2c^2}{2p_{cR}^2} + \frac{1}{2R^2p_{cR}} \frac{\partial}{\partial p_{cR}} [L_{cR}^2 - K_{cR}^2] - \frac{1}{2R^2p_{cR}^2} [L_{cR}^2 - K_{cR}^2] + O(\frac{1}{R^4}) \] (24)

On the right hand side, the first term related to the energy scale (if Gev scale, this term \( \sim 10^{-9} \)), and always negative, i.e. in Einstein theory, no superluminal particles. If the sum of second term and third term is positive, then, at high energy level, there will exist superluminal particle in de Sitter space, unfortunately, after a careful check, this is not the case.

Let assume the particle moves in \( x_1 \) direction, then \( p = p_1 = \dot{x}_1 \neq 0 \), we get

\[ \frac{v - c}{c} = -\frac{m_0^2c^2}{2p_{cR}^2} + \frac{1}{2R^2} \left[ \frac{1}{2} (x_2^2 + x_3^2 - x_0^2) + \frac{1}{2} (x_1^2 + x_2^2 + x_3^2) \frac{p_0^2}{p^2} \right] \] (25)

From \( p^2 = m_0^2 \Gamma^2 x^1 \), \( p_0^2 = m_0^2 \Gamma^2 c^2 \), and if the particle moves in \( x^1 \) direction just on \( x - \) axes, i.e. we can chose \( x_2 = x_3 = 0 \), and \( x = x_1 \) we finally have:

\[ \frac{v - c}{c} = -\frac{m_0^2c^2}{2p_{cR}^2} + \frac{1}{2R^2} [ -x_0^2 + x^2 \cdot \frac{c^2}{v^2} ] \] (26)
As we know $x_0 = ct$, and $x = vt$, then we have the same as in Einstein relativity. So, there is no superluminal particles in 4–d de Sitter space time.

Prof. Yan Mulin told me recently in Hangzhou, $x = a_0 + vt$, then one can get

$$\frac{v - c}{c} \approx -\frac{m_0^2 c^2}{2p_{cR}^2} + \frac{a_0^2}{2R^2} \approx \frac{a_0^2}{2R^2}$$

take $a_0 = 13.7\text{Gly}$, and use the result of OPERA,i.e. $\frac{v - c}{c} = (2.48 \pm 0.28 \pm 0.30)10^{-5}$, Yan got $R \approx 19.5\text{Gl.y.}$
3, Snyders quantized space-time

3.1, The generators of LZG transformation
In the $dS$ spacetime we introduced the generators of LZG transformations, or the Killing vectors, they read

$$
\hat{P}_\mu = -i\hbar(\delta_\mu^\nu - R^{-2}x_\mu x_\nu) \frac{\partial}{\partial x_\nu}, \quad x_\mu := \eta_{\mu\nu}x_\nu; \\
\hat{L}_{\mu\nu} = x_\mu \hat{P}_\nu - x_\nu \hat{P}_\mu = -i\hbar[x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu}] \in so(1,3),
$$

(27)

here in this section the symbol under a hat is marked for the Killing vector. They satisfy

$$
[\hat{P}_\mu, \hat{P}_\nu] = R^{-2} \hat{L}_{\mu\nu}, \\
[\hat{L}_{\mu\nu}, \hat{P}_\rho] = \eta_{\nu\rho} \hat{P}_\mu - \eta_{\mu\rho} \hat{P}_\nu, \\
[\hat{L}_{\mu\nu}, \hat{L}_{\rho\sigma}] = \eta_{\nu\rho} \hat{L}_{\mu\sigma} - \eta_{\nu\sigma} \hat{L}_{\mu\rho} + \eta_{\mu\sigma} \hat{L}_{\nu\rho} - \eta_{\mu\rho} \hat{L}_{\nu\sigma}.
$$

which forms an $so(1,4)$ algebra.

3.2, Snyders quantized space-time
Snyder considered a homogenous quadratic form

$$
-(\frac{\hbar}{a})^2 = \eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2 := \eta^{AB} \eta_A \eta_B < 0,
$$

$$
(\eta^{AB})_{A,B=0,\ldots,4} = (\eta_{AB}) = diag(1, -1, -1, -1, -1),
$$

where $\eta_A$ may be regarded as the homogeneous (projective) coordinates of a real 4-d space of constant curvature, a dS-space. According to Snyder, the quantized space-time coordinators are defined as:

$$
\hat{x}_i = ia(\eta_4 \frac{\partial}{\partial \eta_i} - \eta_i \frac{\partial}{\partial \eta_4})
$$
\[ \hat{t}_i = i \frac{a}{c} (\eta_4 \frac{\partial}{\partial \eta_0} + \eta_0 \frac{\partial}{\partial \eta_4}) \]

Then Snyder defined the energy-momentum with a natural unit of length \( a \)

\[ p_0 = \frac{\hbar}{a} \eta_0, \]

\[ p_\alpha := \frac{\hbar}{a} \eta_\alpha, \quad \hbar = 1, \quad \alpha = 1, 2, 3. \]

\( \hat{t}, \hat{x}_\alpha \) and momenta have the relation as:

\[ \hat{x}_\alpha := i \hbar \left[ \frac{\partial}{\partial p_\alpha} + a^2 p_\alpha p_j \frac{\partial}{\partial p_j} \right], \quad j = 0, \ldots, 3, \]

\[ \hat{x}_0 := i \hbar c \left[ \frac{\partial}{\partial p_0} - \left( \frac{a}{\hbar} \right)^2 p_0 p_j \frac{\partial}{\partial p_j} \right], \quad x_0 = c \hat{t}. \]

They are no longer commutative rather noncommutative ‘quantized’ operators. According to Snyder, they form an \( so(1,4) \) algebra together with the ‘boost’ \( M_\alpha \) and ‘3-angular momentum’ \( L_\alpha \) in the space of momenta:

\[ [\hat{x}_\alpha, \hat{x}_\beta] = ia^2 \hat{L}_\gamma, \]

\[ [\hat{t}, \hat{x}_\alpha] = ia^2 \hat{M}_\alpha, \]

\[ [\hat{L}_\alpha, \hat{L}_\beta] = \epsilon_{\alpha\beta\gamma} \hat{L}_\gamma, \]

\[ [\hat{M}_\alpha, \hat{M}_\beta] = \epsilon_{\alpha\beta\gamma} \hat{M}_\gamma; \quad etc. \]

Here \( \hat{L}_\alpha = \hat{x}_\beta p_\gamma - \hat{x}_\gamma p_\beta \), \( \hat{M}_\alpha = \hat{x}_\alpha p_0 + \hat{x}_0 p_\alpha \).

Obviously, the spacetime no more commute, and the eigenvalue of the coordinate operator \( \hat{x} \) is \( a \), which is a planck scale constant \( (a \sim \ell_P) \).
3.3, Baltrami de Sitter spacetime and Snyders quantized space-time

There is an interchangeable dual relation between Snyder’s model and dS-invariant SR in BdS:

<table>
<thead>
<tr>
<th>Snyder’s QST</th>
<th>dS-invariant SR</th>
</tr>
</thead>
<tbody>
<tr>
<td>momentum ‘picture’</td>
<td>coordinate ‘picture’</td>
</tr>
<tr>
<td>BdS-space of momenta</td>
<td>BdS-spacetime</td>
</tr>
<tr>
<td>$a \sim \text{Planck length}$</td>
<td>$R \sim \text{cosmic radius}$</td>
</tr>
<tr>
<td>quantized space-time</td>
<td>‘quantized’ momenta</td>
</tr>
<tr>
<td>$\hat{x}_\alpha, \hat{t}$</td>
<td>$\hat{p}_\alpha, \hat{E}$</td>
</tr>
</tbody>
</table>

It is important that from two fundamental constants, the Planck length $\ell_P := (G\hbar c^{-3})^{1/2}$ and the $dS$-radius $R \simeq (3/\Lambda)^{1/2}$, it follows a dimensionless constant

$$g := \sqrt{3\ell_P/R}, \quad g^2 \simeq G\hbar c^{-3}\Lambda \sim 10^{-122}. \quad (28)$$

Since there is Newton constant, $g$ should describe the gravity and its self-interaction with local $dS$-invariance between these two scales.

Thus, these indicate that there should be a Planck scale-cosmological constant duality: The cosmological constant is a fundamental scale as the Planck length. The physics at such two scales should be dual to each other in some ‘phase’ space and linked via the gravity with local $dS$-invariance characterized by the dimensionless constant $g$. 
4, Yang’s model

4.1, Yang’s model  C N Yang extended Snyder’s model to the one with the third constant, the radius $R$ of a dS universe in order to recover the translation under $R \to \infty$. Yang found an $so(1,5)$ algebra with $c, a$ and $R$ in a 6-d space with Minkowski ($Mink$) signature. In Yang’s algebra, there are two $so(1,4)$ subalgebras for coordinate operators $\hat{x}^\mu$ and momentum operators $\hat{p}^\mu$, respectively, with a common $so(1,3)$ for angular momentum operators $\hat{l}^{\mu\nu}$. And the algebra is invariant under a $Z_2$ dual transformation between $a, \hat{x}^\mu$ and $\hbar/R, \hat{p}^\mu$. This is a UV-IR parameters’ transformation.

Under Yang’s $so(1,5)$ algebra, there is an invariant quadratic form a 6-d dimensionless $Mink$-space $M^{1,5}$. Then, the metric in $M^{1,5}$ reads

$$d\chi^2 = \eta_{AB} d\zeta^A d\zeta^B, \quad A, B = 0, \ldots, 5,$$

where $\eta_{AB} = \text{diag}(+, -, -, -, -, -)$.

The following operators are just the operators in Yang’s
model up to some redefined coefficients

\[
\hat{x}_0 = ia \left( \zeta^5 \frac{\partial}{\partial \zeta^0} + \zeta^0 \frac{\partial}{\partial \zeta^5} \right), \\
\hat{x}_i = ia \left( \zeta^5 \frac{\partial}{\partial \zeta^i} - \zeta^i \frac{\partial}{\partial \zeta^5} \right), \\
\hat{p}_0 = \frac{i\hbar}{R} \left( \zeta^4 \frac{\partial}{\partial \zeta^0} + \zeta^0 \frac{\partial}{\partial \zeta^4} \right), \\
\hat{p}_i = \frac{i\hbar}{R} \left( \zeta^4 \frac{\partial}{\partial \zeta^i} - \zeta^i \frac{\partial}{\partial \zeta^4} \right), \\
\hat{M}_i = i\hbar \left( \zeta^0 \frac{\partial}{\partial \zeta^i} + \zeta^i \frac{\partial}{\partial \zeta^0} \right), \\
\hat{L}_i = i\hbar \epsilon_{ij} \left( \frac{\partial}{\partial \zeta^k} \right), \\
\hat{\psi} = ia R \left( \zeta^5 \frac{\partial}{\partial \zeta^0} - \zeta^0 \frac{\partial}{\partial \zeta^5} \right)
\]

with \( \epsilon_{123} = \epsilon_1^{23} = 1 \) and \( \zeta_j = \eta_{jA} \zeta^A \). Define \( \hat{\mu}^{\nu} = i\hbar (\zeta^{\mu} \frac{\partial}{\partial \zeta^\nu} - \zeta^{\nu} \frac{\partial}{\partial \zeta^\mu}) \)

Then Yang’s \( so(1, 5) \) algebra is given as follows:

\[
[\hat{\mu}^{\nu}, \hat{\nu}^{\rho}] = i\hbar R^{-2} \hat{l}^{\mu\nu}, \quad [\hat{\mu}^{\nu}, \hat{\rho}^{\rho}] = i\hbar (\eta^{\mu\rho} \hat{\nu}^{\rho} - \eta^{\mu\rho} \hat{\rho}^{\nu}), \\
[\hat{x}^{\mu}, \hat{x}^{\nu}] = i\hbar a^2 \hat{\mu}^{\nu}, \quad [\hat{\mu}^{\nu}, \hat{\rho}^{\rho}] = i\hbar (\eta^{\mu\rho} \hat{x}^{\nu} - \eta^{\mu\rho} \hat{x}^{\nu}), \\
[\hat{x}^{\mu}, \hat{\rho}^{\nu}] = i\hbar \eta^{\mu\nu} \hat{\psi}, \quad [\hat{\psi}, \hat{\mu}^{\nu}] = -ia^2 \hbar^{-1} \hat{p}^{\mu}, \\
[\hat{\psi}, \hat{\rho}^{\mu}] = i\hbar R^{-2} \hat{x}^{\mu}, \quad [\hat{\psi}, \hat{l}^{\mu\nu}] = 0,
\]

\[
\hat{x}^{\mu} = \eta^{\mu\nu} \hat{x}_{\nu}, \quad \hat{p}^{\mu} = \eta^{\mu\nu} \hat{p}_{\nu},
\]

together with an \( so(1, 3) \) for the 4-d angular momentum operators.

It is clear that there are two \( so(1, 4) \) for coordinate operators \( \hat{x}^{\mu} \) and momentum operators \( \hat{p}^{\nu} \), respectively, with a common \( so(1, 3) \) for \( \hat{l}^{\mu\nu} \). It is also clear that in Yang’s algebra with respect to the 6-d ‘angular momentum’ there is a \( Z_2 = \{ e, r \mid r^2 = e \} \) dual transformation with

\[
r: \quad a \rightarrow \frac{\hbar}{R}, \quad \hat{x}^{\mu} \rightarrow \hat{p}^{\mu}, \quad \hat{\psi} \rightarrow -\hat{\psi}.
\]
Since $a$ is near or equal to the Planck length $\ell_P$ and $R$ is the radius of a dS universe, the invariance under the $Z_2$ dual transformation is a UV-IR duality.

### 4.2, From Yang’s model to Snyder’s model and BdS model

Let’s consider a dimensionless 5-d de Sitter space in 6–Mink space:

$$dS_5 : \quad \eta_{AB} \zeta^A \zeta^B = -\frac{R^2}{a^2}$$

On the $\zeta^4 = 0$ intersection of this $dS_5$, introduce dimensional coordinates:

$$\eta_\mu = \frac{\hbar}{R} \zeta^\mu, \quad \eta_4 = \frac{\hbar}{R} \zeta^5,$$

then we get

$$\eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2 = -\frac{\hbar^2}{a^2}.$$

This is just Snyder’s 4–d momentum de Sitter space.

On the other hand, the $\zeta^5 = 0$ intersection of this $dS_5$, we can introduce dimensional coordinates:

$$\xi_\mu = a \zeta^\mu, \quad \xi_4 = a \zeta^4,$$

then we get

$$\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 - \xi_4^2 = -R^2.$$

This is just 4–d de Sitter space of BdS model.

So Yang’s model contains all information from ultra-micro scope to cosmic scope, where we may define special relativity with three constants: $a$, $R$ and $c$. 
Conclusions

• 1, In the cosmic scale, the Einstein’s special relativity ($SR_c$) should extend to 4-d de Sitter spacetime relativity ($SR_{CR}$) which returns to $SR_c$ when $R \to \infty$.

• 2, In the de Sitter special relativity ($SR_{CR}$), superluminal particle are not allowed.

• 3, In ultra-microscope scale (Planck scale), spacetime should be quantized, and Snyder’s 4-d momentum de-Sitter space need to be introduced.

• 4, Complete Yang model is a unification theory for 4-d de Sitter spacetime relativity ($SR_{CR}$) and Snyder’s quantized spacetime theory (SQT); Yang model (from $dS_5$), can be used as a more general ”phase space” for universal scale.

• 5, finally, the complete Yang model should be regarded as a theory of the $SR$ based on the principle of inertia in the both spacetime and space of momentum as well as the postulate on three universal constants $c, \ell_P$ and $R$. 
THANKS!!!