Dense Baryonic Matter in the Hidden Local Symmetry Approach

Skyrmions, half-skyrmions and nucleon mass

Yong-Liang Ma

Department of Physics, Nagoya University.

Seminar given @ IHEP, Beijing, May 21, 2013.
**Purpose:**

Baryonic matter using Skyrme model

**Advantages:** Both single skyrmion and skyrmion matter can be unifiedly described;

It’s a natural framework to investigate the in-medium modification of hadron properties. Chiral symmetry restoration, nucleon mass, etc.
References:


In Collaboration with:
We included all the roles of $\pi$, $\rho$ and $\omega$ mesons using the HLS up to $O(p^4)$ including the Wess-Zumino terms.

We use a general master formula to determine all the LECs self-consistently from a class of holographic QCD models.

The baryonic matter was studied by putting the solitons on the FCC crystals.

Hardron properties in-medium were investigated by considering the meson fluctuations with respect to the classical solutions.
CONTENTS

I. Introduction

II. Baryon properties in the Hidden Local Symmetry induced from holographic models

III. Dense baryonic matter from HLS

IV. Hadron properties with the FCC crystal background

V. Summary and discussions
I. Introduction

Skyrme Model

1960s: T.H.R. Skyrme

Baryons are topological solitons in a nonlinear theory of pions

\[ \mathcal{L} = \frac{f_\pi^2}{4} \text{Tr} (\partial_\mu U^\dagger \partial^\mu U) + \frac{1}{32e^2} \text{Tr} \left[ U^\dagger \partial_\mu U, U^\dagger \partial_\nu U \right]^2 \]

- $f_\pi$: pion decay constant
- $e$: Skyrme parameter

Topological soliton
winding number = baryon number

\[ B_\mu = \frac{1}{24\pi^2} \epsilon_{\mu\nu\alpha\beta} \text{Tr} \left( U^\dagger \partial_\nu U U^\dagger \partial_\alpha U U^\dagger \partial_\beta U \right) \]


M. Harada and YM, "Lecture notes on the Skyrme model".
In large $N_c$, effective field theory of mesons is a weakly coupled theory and baryons may be emerge as solitons in this theory.

**E. Witten 1980s**

**Hedgehog Solution**

\[ U = \exp (iF(r) \tau \cdot \hat{r}) \]

\[ R \sim 1 \text{ fm} \]
\[ M_{\text{sol}} \sim 146 |B| \left( \frac{f_\pi}{2e} \right) \sim 1.2 \text{ GeV} \]
for $B = 1$

\[ M_{\text{sol}} \sim 1.23 \times 12 \pi^2 |B| > 12 \pi^2 |B| \]

in the Skyrme unit: $f_\pi/(2e)$

Bogomolny bound
Baryon masses

- To give correct quantum numbers
- SU(2) collective coordinate quantization
  \[ U(t) = A(t)U_0A^\dagger(t) \]
- Mass formula: infinite tower of \( I = J \)
  \[ M = M_{\text{sol}} + \frac{1}{2I}I(I+1) \quad I : \text{moment of inertia} \]
  \[ M_N = M_{\text{sol}} + \frac{3}{8I}, \quad M_\Delta = M_{\text{sol}} + \frac{15}{8I} \]
- Adjust \( f_\pi \) and \( e \) to reproduce the nucleon and Delta masses
  \[ f_\pi = 64.5 \text{ MeV}, \quad e = 5.45 \]
  Empirically, \( f_\pi = 93 \text{ MeV}, \quad e = 5.85(?) \)
Skyrme model: results

- Adding the pion-mass term

\[ \mathcal{L}_{\text{pion}} = \frac{1}{2} m_{\pi}^2 f_{\pi}^2 (\text{Tr}(U) - 2) \]

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Prediction (massless pion)</th>
<th>Prediction (massive pion)</th>
<th>Expt</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_N )</td>
<td>input</td>
<td>input</td>
<td>939 MeV</td>
</tr>
<tr>
<td>( M_\Delta )</td>
<td>input</td>
<td>input</td>
<td>1232 MeV</td>
</tr>
<tr>
<td>( f_{\pi} )</td>
<td>64.5 MeV</td>
<td>54 MeV</td>
<td>93 MeV</td>
</tr>
<tr>
<td>( \langle r^2 \rangle_{I=0}^{1/2} )</td>
<td>0.59 fm</td>
<td>0.68 fm</td>
<td>0.72 fm</td>
</tr>
<tr>
<td>( \langle r^2 \rangle_{I=1}^{1/2} )</td>
<td>( \infty )</td>
<td>1.04 fm</td>
<td>0.88 fm</td>
</tr>
<tr>
<td>( \langle r^2 \rangle_{M,I=0}^{1/2} )</td>
<td>0.92 fm</td>
<td>0.95 fm</td>
<td>0.81 fm</td>
</tr>
<tr>
<td>( \langle r^2 \rangle_{M,I=1}^{1/2} )</td>
<td>( \infty )</td>
<td>1.04 fm</td>
<td>0.80 fm</td>
</tr>
<tr>
<td>( \mu_p )</td>
<td>1.87</td>
<td>1.97</td>
<td>2.79</td>
</tr>
<tr>
<td>( \mu_n )</td>
<td>-1.31</td>
<td>-1.24</td>
<td>-1.91</td>
</tr>
<tr>
<td>(</td>
<td>\mu_p/\mu_n</td>
<td>)</td>
<td>1.43</td>
</tr>
</tbody>
</table>

Adkins, Nappi, Witten, NPB 228 (1983).
Skyrmion: A classical object!

Easy to construct many-body system
Two skyrmions

1960, T. H. R. Skyrme
Skyrmion crystal (Cubic)

Klebanov symmetry of the ground state does not yield the lowest energy
Skyrmion crystal (FCC)

Chiral Symmetry Restoration?

- single skyrmion FCC $\langle \text{tr}(U) \rangle \neq 0$
- half-skyrmion CC $\langle \text{tr}(U) \rangle = 0$

\[ L_F \]
\[ \langle \text{tr}(U) \rangle \]
Classical solution of the meson Lagrangian!

In-medium properties of the mesons
\[ \mathcal{L} = \frac{f_{\pi}^2}{4} \text{Tr}(\partial_{\mu} U^+ \partial^\mu U) + \frac{1}{32e^2} \text{Tr} \left[ U^+ \partial_{\mu} U, U^+ \partial_{\nu} U \right]^2 \]

\[ U_{\pi} = \exp(i \vec{\pi} \cdot \vec{\pi}) \]

\[ \mathcal{L} = \frac{1}{2} \partial_{\mu} \vec{\pi} \cdot \partial^\mu \vec{\pi} + \ldots \]
\[ \mathcal{L} = \frac{f^2_\pi}{4} \text{Tr}(\partial_\mu U^\dagger \partial^\mu U) + \frac{1}{32e^2} \text{Tr} \left[ U^\dagger \partial_\mu U, U^\dagger \partial_\nu U \right]^2 \]

\[ U = \sqrt{U_\pi U_B} \sqrt{U_\pi} \]

\[ \mathcal{L} = \frac{1}{2} G_{ab}(U_B) \partial_\mu \pi^a \partial^\mu \pi^b + \cdots \]

\[ \frac{f^*_\pi}{f_\pi} = \sqrt{\langle G_{aa}(U_B) \rangle} \]
Skyrme model for Hadron Physics

**Single baryon**

**Improvement of the model**
- More meson degrees of freedom
- Higher order interactions
- Corrections from $1/N_c$ expansion

**Extensions to other hadrons**
- SU(3) extension, hyperons
- Heavy quark baryons
- Hypernuclei & exotic baryons

**Nucleon matter**

**Topics**
- In-medium properties of single baryon
- Equation of State
- Phase transition
- Application to nuclei

**Approaches**
- Modified effective Lagrangian
- Skyrmion crystal
- Winding number $n$ solutions

Still many works to do
Going up $\infty$ tower
Early attempts to include vector mesons

Including $\omega$ meson

$$\mathcal{L} = \mathcal{L}_{\text{pion}} + \mathcal{L}_{\omega} + \mathcal{L}_{\text{int}}$$

$$\mathcal{L}_{\text{pion}} = \frac{f_\pi^2}{4} \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \frac{f_\pi^2}{2} m_\omega^2 (\text{Tr}(U) - 2)$$

$$\mathcal{L}_{\omega} = \frac{m_\omega^2}{2} \omega_\mu \omega^\mu - \frac{1}{4} \omega_{\mu\nu} \omega^{\mu\nu}$$

\[\mathcal{L}_{\text{int}} = \beta \omega_\mu B^\mu\]


Including $\rho$ meson

$$\mathcal{L} = \mathcal{L}_{\text{pion}} + \mathcal{L}_{\rho} + \mathcal{L}_{\text{int}}$$

\[\mathcal{L}_{\text{int}} = \alpha \text{Tr}(\rho_{\mu\nu} \partial^\mu U^\dagger U \partial^\nu U^\dagger)\]


<table>
<thead>
<tr>
<th>Quantity</th>
<th>Skyrme (massive pion)</th>
<th>$\omega$</th>
<th>$\rho$</th>
<th>Expt</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_N$</td>
<td>input</td>
<td>input</td>
<td>input</td>
<td>939 MeV</td>
</tr>
<tr>
<td>$M_\Delta$</td>
<td>input</td>
<td>input</td>
<td>input</td>
<td>1232 MeV</td>
</tr>
<tr>
<td>$f_\pi$</td>
<td>54 MeV</td>
<td>62 MeV</td>
<td>52.4 MeV</td>
<td>93 MeV</td>
</tr>
<tr>
<td>$\langle r^2 \rangle_{I=0}^{1/2}$</td>
<td>0.68 fm</td>
<td>0.74 fm</td>
<td>0.70 fm</td>
<td>0.72 fm</td>
</tr>
<tr>
<td>$\langle r^2 \rangle_{I=1}^{1/2}$</td>
<td>1.04 fm</td>
<td>1.06 fm</td>
<td>1.08 fm</td>
<td>0.88 fm</td>
</tr>
<tr>
<td>$\langle r^2 \rangle_{M,I=0}^{1/2}$</td>
<td>0.95 fm</td>
<td>0.92 fm</td>
<td>0.98 fm</td>
<td>0.81 fm</td>
</tr>
<tr>
<td>$\langle r^2 \rangle_{M,I=1}^{1/2}$</td>
<td>1.04 fm</td>
<td>1.02 fm</td>
<td>1.06 fm</td>
<td>0.80 fm</td>
</tr>
<tr>
<td>$\mu_p$</td>
<td>1.97</td>
<td>2.34</td>
<td>2.16</td>
<td>2.79</td>
</tr>
<tr>
<td>$\mu_n$</td>
<td>-1.24</td>
<td>-1.46</td>
<td>-1.38</td>
<td>-1.91</td>
</tr>
<tr>
<td>$</td>
<td>\mu_p/\mu_n</td>
<td>$</td>
<td>1.59</td>
<td>1.60</td>
</tr>
<tr>
<td>$\mu_{I=0}$</td>
<td>0.365</td>
<td>0.44</td>
<td>0.39</td>
<td>0.44</td>
</tr>
<tr>
<td>$\mu_{I=1}$</td>
<td>1.605</td>
<td>1.9</td>
<td>1.77</td>
<td>2.35</td>
</tr>
</tbody>
</table>
Vector mesons

- Systematic way to include vector mesons
  - Massive Yang-Mills approach: Syracuse group
  - Hidden Local Symmetry: Nagoya group
  - Equivalence of the two approaches

- Skyrmions in the HLS
  - \( \rho \) meson stabilized model
  - \( \rho \) and \( \omega \) meson stabilized model
  - \( \rho, \omega \) and \( a_1 \) meson stabilized model

- Reviews
Earlier works

\( \mathcal{O}(p^2) \) Lagrangian with HLS

\[
\mathcal{L}_\sigma = \frac{f_\pi^2}{4} \text{Tr}(\partial_\mu U \partial^\mu U^\dagger) \quad \text{with} \quad U = \xi_L^\dagger \xi_R
\]

Hidden Symmetry

\[
\xi_{L,R}(x) \rightarrow h(x)\xi_{L,R}(x), \quad h \in \text{SU}(2)
\]

\[
V_{\mu}(x) \rightarrow ih(x)\partial_\mu h^\dagger(x) + h(x)V_{\mu}(x)h^\dagger(x)
\]

Covariant derivative:

\[
D_\mu \xi_{L,R} = \partial_\mu \xi_{L,R} - iV_\mu \xi_{L,R}
\]

\[
\hat{\alpha}_{\mu \parallel} = \frac{1}{2i}(D_\mu \xi_L \xi_L^\dagger + D_\mu \xi_R \xi_R^\dagger)
\]

\[
\hat{\alpha}_{\mu \perp} = \frac{1}{2i}(D_\mu \xi_L \xi_L^\dagger - D_\mu \xi_R \xi_R^\dagger)
\]

Unitary gauge: \( \xi_L^\dagger = \xi_R = \xi \)

HLS Lagrangian

\[
\mathcal{L} = \mathcal{L}_A + a \mathcal{L}_V + \mathcal{L}_{\text{kin}}
\]

\[
\mathcal{L}_A = f_\pi^2 \text{Tr}(\hat{\alpha}_{\mu \perp}^2) = \mathcal{L}_\sigma, \quad \mathcal{L}_V = f_\pi^2 \text{Tr}(\hat{\alpha}_{\mu \parallel}^2)
\]

\[
\mathcal{L}_{\text{kin}} = -\frac{1}{2g^2} \text{Tr}(F_{\mu \nu}^2)
\]

\[
m_V^2 = ag^2 f_\pi^2
\]

\[
g_{\rho \pi \pi} = \frac{1}{2}ag
\]

\( a = 2 \) gives KSRF relation and the universality of \( \rho \) coupling.

$\rho$ meson and the Skyrme term

As $a \to \infty$, i.e., as $m_V \to \infty$

$$\mathcal{L}_V \propto (\alpha_{\mu||} - V_\mu)^2 = 0$$

where

$$\alpha_{\mu||} = \frac{1}{2i} \left( \partial_\mu \xi_L \xi_L^\dagger + \partial_\mu \xi_R \xi_R^\dagger \right)$$

$$\Rightarrow$$

$$\mathcal{L}_{\text{kin}} \to \frac{1}{32g^2} \text{Tr} \left[ \partial_\mu UU^\dagger, \partial_\nu UU^\dagger \right]^2 = \mathcal{L}_{\text{Skyrme}}$$

Skyrmion in the HLS with the $\rho$ meson

$$M_{\text{sol}} = (667 \sim 1575) \text{ MeV}$$

for $1 \leq a \leq 4$

$$M_{\text{sol}} = 1045 \text{ MeV} \quad \text{for } a = 2$$

---

ρ and ω mesons

ω meson: introduced through HGS like the ρ meson

Anomalous Lagrangian: source of the ω meson

\[ \mathcal{L}_{an} = \frac{3}{8} g N_c (c_1 - c_2 - c_3) \omega_\mu B^\mu \]

\[ - \frac{g^3 N_c}{32 \pi^2} (c_1 + c_2) \varepsilon^{\mu\nu\alpha\beta} \omega_\mu \text{tr} (a_\nu \bar{\rho}_\alpha \bar{\rho}_\beta) \]

\[ - \frac{g N_c}{8 \pi^2} c_3 \varepsilon^{\mu\nu\alpha\beta} \left\{ - \omega_\mu \text{tr} (a_\nu v_\alpha v_\beta) + \frac{i g}{4} \partial_\mu \omega_\nu \text{tr} (a_\alpha \rho_\beta - \rho_\alpha a_\beta) - \frac{i g}{4} \omega_\mu \text{tr} (\rho_\nu a_\beta) \right\} , \]

Determination of parameters

Minimal model: \[ c_1 = \frac{2}{3}, c_2 = -\frac{2}{3}, c_3 = 0 \] \( \omega^\mu B_\mu \) term only

Vector Dominance: \[ c_1 = 1, c_2 = 0, c_3 = 1 \] \( \omega \pi^3 \) term

Or fit them to known phenomenology

See, for example, P. Jain, U.-G. Meissner, N. Kaiser, H. Weigel, N.C. Mukhopadhyay, etc


minimal model results with \( a = 2, f_\pi = 93 \, \text{MeV}, \, g = 5.85 \)

\[ M_{\text{sol}} = 1475 \, \text{MeV} \]

\[ \begin{align*}
\begin{array}{c}
\text{Minimal model} \\
\text{Complete} \\
\text{Following} \\
\text{ref.}^{17)}
\end{array}
\end{align*} \]

\[ \begin{array}{c|c|c|c}
& \text{Minimal model} & \text{Complete model} & \text{Following} \\
\hline
M_H [\text{MeV}] & 1474 & 1465 & 1057 \\
r_H [\text{fm}] & 0.50 & 0.48 & 0.27 \\
\hline
\end{array} \]

For comparison, the results of the model of ref.\(^{17)}\) including pions and \( \rho \) mesons are also given. The parameters used are \( m_\pi = 139 \, \text{MeV}, \, f_\pi = 93 \, \text{MeV}, \, \text{and} \, g = 5.85 \). Here \( M_H \) is the static soliton mass, and \( r_H \) the baryonic r.m.s. radius.
\( \rho, \omega \) and \( a_1 \) mesons

- Axial vector meson
  \[
  U(x) = \xi_L^\dagger(x)\xi_M(x)\xi_R(x)
  \]

- 14 anomalous terms
  cf. 6 independent terms in the \( \pi \rho \omega \) system

- Hard to control the parameters

  \textit{Results with } a = 2, f_\pi = 93 \text{ MeV, } g = g_\omega / 1.5 = 5.85, m_V = 770 \text{ MeV}
  \[
  M_{\text{sol}} = 1002 \text{ MeV}
  \]

---

\textit{Fig. 1. The behaviour of the skyrmion energy as the vector meson couplings and the masses go to zero. (a) } g = g_\omega / 1.5 \rightarrow 0, (b) g \rightarrow 0, g_\omega / 1.5 = 5.85 \text{ (fixed), (c) } g_\omega \rightarrow 0, g = 5.85 \text{ (fixed). In all cases the ratios } g/m \text{ and } g_\omega / 1.5 m \text{ are kept constant at } 5.85 / 770 \text{ MeV.}

The results are sensitive to the parameters!
Problems in the previous works

1. Parameter $a$ dependence.
   - Ambiguity in the value of $a$ results in a large uncertainty in the soliton mass.
     (In free space, $a \approx 2$ and at high temperature/density $a \approx 1$)
   - Hinders systematic study of baryonic matter

2. Higher order terms.
   - Higher order terms, such as $O(p^4)$ terms, are at $O(N_c)$ like $O(p^2)$; should be considered.
   - More complicated form of the Lagrangian
   - Uncontrollable large number of low energy constants.

In our work:
- Full $O(p^4)$ terms with $\pi$, $\rho$, and $\omega$ mesons
- Fix all LECs by using hQCD models
Skyrmion physics from hQCD models

- **Holographic QCD**: infinite tower of vector mesons Solitons in hQCD:

- Skyrmions in HLS with ρ meson up to $O(p^4)$ terms with hQCD:

- Effect of the tower structure in a 5-d conformal Yang-Mills theory:
  - Paul Sutcliffe, JHEP 1104 (2011) 045.
The role of the infinite tower of vector mesons in the baryon structure can be studied in the approximation that the space is flat and the CS term is ignored. The BPS Skyrmion

Consider the 5D Euclidean YM action

\[ S = -\frac{1}{2} \int \text{tr} F_{mn}^2 d^4x \, dz, \]

where

\[ A_m = T^a A^a_m \]

\[ F_{mn} = \partial_m A_n - \partial_n A_m + [A_m, A_n] \]

\[ F_{MN} = * F_{MN} \]

\[ B = \frac{1}{16\pi^2} \int \text{tr}(F_{MN}^* F_{MN}) d^3x \, dz \]
The more vector mesons are included, the closer the static energy goes down and approaches the BPS bound. In other words, the higher tower of vector mesons drive the theory to a conformal theory.

\[ E^{(0)} = 1.235 (8\pi^2 B). \]

\[ E^{(1)} = 1.071 (8\pi^2 B) \]

\[ E^{(2)} = 1.048 (8\pi^2 B). \]

The full tower will bring this to the bound \( E^{(\infty)} = 8\pi^2 B \)

The high-lying vector mesons make the theory flow to a conformal theory.
II. Baryon Properties in the Hidden Local Symmetry Induced from Holographic Models


\[ \mathcal{L}_{\text{HLS}} = \mathcal{L}_{(2)} + \mathcal{L}_{(4)} + \mathcal{L}_{\text{anom}}, \]
\[ \mathcal{L}_{(2)} = f_\pi^2 \text{Tr} \left( \hat{a}_{\perp \mu} \hat{a}_{\perp}^{\mu} \right) + a f_\pi^2 \text{Tr} \left( \hat{a}_{|| \mu} \hat{a}_{||}^{\mu} \right) - \frac{1}{2 g^2} \text{Tr} \left( V_{\mu \nu} V^{\mu \nu} \right) \]
\[ \mathcal{L}_{(4)} = \mathcal{L}_{(4)y} + \mathcal{L}_{(4)z} \]
\[ \mathcal{L}_{(4)y} = y_1 \text{Tr} \left[ \hat{a}_{\perp \mu} \hat{a}_{\perp}^{\mu} \hat{a}_{\perp \nu} \hat{a}_{\perp}^{\nu} \right] + y_2 \text{Tr} \left[ \hat{a}_{\perp \mu} \hat{a}_{\perp \nu} \hat{a}_{\perp}^{\mu} \hat{a}_{\perp}^{\nu} \right] + y_3 \text{Tr} \left[ \hat{a}_{|| \mu} \hat{a}_{||}^{\mu} \hat{a}_{|| \nu} \hat{a}_{||}^{\nu} \right] + y_4 \text{Tr} \left[ \hat{a}_{|| \mu} \hat{a}_{|| \nu} \hat{a}_{\perp}^{\mu} \hat{a}_{\perp}^{\nu} \right] \]
\[ + y_5 \text{Tr} \left[ \hat{a}_{\perp \mu} \hat{a}_{\perp}^{\mu} \hat{a}_{|| \nu} \hat{a}_{||}^{\nu} \right] + y_6 \text{Tr} \left[ \hat{a}_{\perp \mu} \hat{a}_{\perp \nu} \hat{a}_{||}^{\mu} \hat{a}_{||}^{\nu} \right] + y_7 \text{Tr} \left[ \hat{a}_{\perp \mu} \hat{a}_{\perp \nu} \hat{a}_{\perp}^{\mu} \hat{a}_{||}^{\nu} \right] \]
\[ + y_8 \left\{ \text{Tr} \left[ \hat{a}_{\perp \mu} \hat{a}_{\perp}^{\mu} \hat{a}_{\perp \nu} \hat{a}_{\perp}^{\nu} \right] + \text{Tr} \left[ \hat{a}_{\perp \mu} \hat{a}_{|| \nu} \hat{a}_{\perp}^{\mu} \hat{a}_{\perp}^{\nu} \right] \right\} + y_9 \text{Tr} \left[ \hat{a}_{\perp \mu} \hat{a}_{|| \nu} \hat{a}_{\perp}^{\mu} \hat{a}_{\perp}^{\nu} \right], \]
\[ \mathcal{L}_{(4)z} = i z_4 \text{Tr} \left[ V_{\mu \nu} \hat{a}_{\perp \mu}^{\nu} \hat{a}_{\perp \nu} \right] + i z_5 \text{Tr} \left[ V_{\mu \nu} \hat{a}_{|| \mu}^{\nu} \hat{a}_{\perp \nu} \right]. \]
\[ \mathcal{L}_{\text{anom}} = \frac{N_c}{16 \pi^2} \sum_{i=1}^{3} c_i \mathcal{L}_i, \]
\[ \mathcal{L}_1 = i \text{Tr} \left[ \hat{a}_{L}^{3} \hat{a}_{R} - \hat{a}_{R}^{3} \hat{a}_{L} \right], \]
\[ \mathcal{L}_2 = i \text{Tr} \left[ \hat{a}_{L} \hat{a}_{R} \hat{a}_{L} \hat{a}_{R} \right], \]
\[ \mathcal{L}_3 = \text{Tr} \left[ F_{\nu} \left( \hat{a}_{L} \hat{a}_{R} - \hat{a}_{R} \hat{a}_{L} \right) \right], \]

17 parameters! Difficult to fix them phenomenologically!
HLS from hQCD models

\[ S_5 = S_5^\text{DBI} + S_5^\text{CS} \]
\[ S_5^\text{DBI} = N_c G_{YM} \int d^4x dz \left\{ -\frac{1}{2} K_1(z) \text{Tr} [\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}] + K_2(z) M_{KK}^2 \text{Tr} [\mathcal{F}_{\mu z} \mathcal{F}^{\mu z}] \right\} \]
\[ S_5^\text{CS} = \frac{N_c}{24\pi^2} \int_{M^4 \times R} w_5(A) \]
\[ w_5(A) = \text{Tr} \left[ A \mathcal{F}^2 + \frac{i}{2} A^3 \mathcal{F} - \frac{1}{10} A^5 \right] \]

A_z = 0 gauge
Mode expansion

\[ A_\mu^{\text{integ}}(x, z) = \hat{\alpha}_{\mu \perp}(x) \psi_0(z) + [\hat{\alpha}_{\mu \parallel}(x) + V_\mu(x)] \]
\[ + \hat{\alpha}_{\mu \parallel}(x) \psi_1(z), \]

\[ -K_1(z) \partial_z [K_2(z) \partial_z \psi_n(z)] = \lambda_n \psi_n(z), \]

16 parameters in terms of 2 parameters: G_{YM}, M_{KK}
Inputs: f_\pi, m_\rho.
\[
(E/B) = (E/B)_{O(p^2)} + (E/B)_{O(p^4)} + (E/B)_{hWZ},
\]

where

\[
(E/B)_{O(p^2)} = \frac{1}{4} \int_{\text{Box}} d^3 x \left[ \frac{f^2}{2} (\tilde{a}_{\perp i} \tilde{a}_{\perp i}) - \frac{m_\omega^2}{2} \omega(r)^2 + \frac{m_\rho^2}{2g^2} (\tilde{a}_{\parallel i} \tilde{a}_{\parallel i}) - \frac{1}{2} \partial_i \omega \partial_i \omega + \frac{1}{g^2} V_{ij} \tilde{a}_{\parallel i} \tilde{a}_{\parallel j} \right],
\]

\[
(E/B)_{O(p^4)} = -\frac{1}{4} \int_{\text{Box}} d^3 x \left[ \frac{g_1}{4} (\tilde{a}_{\perp i} \times \tilde{a}_{\perp j}) \cdot (\tilde{a}_{\perp i} \times \tilde{a}_{\perp j}) + \frac{g_3}{4} (\tilde{a}_{\parallel i} \times \tilde{a}_{\parallel j}) \cdot (\tilde{a}_{\parallel i} \times \tilde{a}_{\parallel j}) + \frac{g_5}{4} \{ (\tilde{a}_{\perp i} \times \tilde{a}_{\parallel j}) \cdot (\tilde{a}_{\perp i} \times \tilde{a}_{\parallel j}) \} \right] + \frac{g_6}{4} (\tilde{a}_{\perp i} \times \tilde{a}_{\perp j}) \cdot (\tilde{a}_{\parallel i} \times \tilde{a}_{\parallel j}) - \frac{z_1}{2} V_{ij} \cdot (\tilde{a}_{\perp i} \times \tilde{a}_{\perp j}) - \frac{z_5}{2} V_{ij} \cdot (\tilde{a}_{\parallel i} \times \tilde{a}_{\parallel j})],
\]

\[
(E/B)_{hWZ} = \frac{1}{4} \left( \frac{gN_c}{2\pi^2} \right) \int_{\text{Box}} d^3 x \left[ -c_1 \omega \epsilon_{ijk} \left[ \tilde{a}_{\perp i} \cdot (\tilde{a}_{\parallel j} \times \tilde{a}_{\parallel k}) + \tilde{a}_{\perp i} \cdot (\tilde{a}_{\parallel j} \times \tilde{a}_{\parallel k}) \right] - c_2 \omega \epsilon_{ijk} \left[ \tilde{a}_{\perp i} \cdot (\tilde{a}_{\parallel j} \times \tilde{a}_{\parallel k}) - \tilde{a}_{\perp i} \cdot (\tilde{a}_{\parallel j} \times \tilde{a}_{\parallel k}) \right] + 2c_3 \epsilon_{ijk} \left[ \partial_i \omega \left( \tilde{a}_{\parallel j} \tilde{a}_{\parallel k} \right) + \omega \left( V_{ij} \tilde{a}_{\parallel k} \right) \right] \right],
\]

\(\omega - \rho - \pi\) interaction

\(\omega - 3\pi\) interaction

Minimal model
In hQCD models, the mass scale $M_{KK}$ and the 't Hooft coupling $G_{YM}$ are free parameters.

$$m_\rho = 775.49 \text{ MeV},$$
$$f_\pi = 92.4 \text{ MeV}.$$

Two hQCD models:

1. Sakai-Sugimoto model: $K_1(z) = K^{-1/3}(z), K_2(z) = K(z)$, with $K(z) = 1 + z^2$;
2. Flat space (BPS): $K_1(z) = K_2(z) = 1$.

<table>
<thead>
<tr>
<th>Model</th>
<th>$y_1$</th>
<th>$y_3$</th>
<th>$y_5$</th>
<th>$y_6$</th>
<th>$z_4$</th>
<th>$z_5$</th>
<th>$c_1$</th>
<th>$c_2$</th>
<th>$c_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SS model</td>
<td>-0.001096</td>
<td>-0.002830</td>
<td>-0.015917</td>
<td>+0.013712</td>
<td>0.010795</td>
<td>-0.007325</td>
<td>+0.381653</td>
<td>-0.129602</td>
<td>0.767374</td>
</tr>
<tr>
<td>BPS model</td>
<td>-0.071910</td>
<td>-0.153511</td>
<td>-0.012286</td>
<td>-0.196545</td>
<td>0.090338</td>
<td>-0.130778</td>
<td>-0.206992</td>
<td>+3.031734</td>
<td>1.470210</td>
</tr>
</tbody>
</table>

TABLE I. Low-energy constants of the HLS Lagrangian at $O(p^4)$ with $a = 2$. 
Skyrme parameter from hQCD models

Original Skyrme model

\[ \mathcal{L}_{Sk} = \frac{f^2}{4} \text{Tr} (\partial_\mu U \partial_\mu U^\dagger) + \frac{1}{32e^2} \text{Tr} [\partial_\mu UU^\dagger, \partial_\nu UU^\dagger]^2 \]

Integrate out all vector modes

\[ \mathcal{L}_{\text{ChPT}} = f^2 \text{Tr} \left[ \alpha_{\perp \mu} \alpha_{\perp}^\mu \right] + \left( \frac{1}{2g^2} - \frac{z_4}{2} - \frac{y_1 - y_2}{4} \right) \text{Tr} \left[ \alpha_{\perp \mu, \alpha_{\perp \nu}} \right]^2 + \frac{y_1 + y_2}{4} \text{Tr} \{\alpha_{\perp \mu, \alpha_{\perp \nu}}\}^2 \]

\[ \frac{1}{2e^2} = \frac{1}{2g^2} - \frac{z_4}{2} - \frac{y_1 - y_2}{4} \]

Both values are larger than that used in the original Skyrme model because of the contributions from \( y_1, y_2, \) and \( z_4 \) terms at \( O(p^4) \). \( \rightarrow O(p^4) \) contributions.

Skyrme parameter value obtained in the at space-time is even larger than that estimated in the SS model. \( \rightarrow \) depends on geometry.
Three versions of HLS:

1. HLS($\pi, \rho, \omega$) model:
   Full $O(p^4)$ Lagrangian with $hWZ$ terms.

2. HLS($\pi, \rho$) model:
   Full $O(p^4)$ Lagrangian without $hWZ$ terms, $\omega$ meson decouples.

3. HLS($\pi$) model:
   Integrate out vector mesons, the same as Skyrme model but $e$ is fixed by $hQCD$ models.
Soliton wave functions

**Classical Solution**

\[
\xi(r) = \exp \left[ i\tau \cdot \hat{r} \frac{F(r)}{2} \right]
\]

\[
\omega_\mu = W(r) \delta_{0 \mu},
\]

\[
\rho_0 = 0, \quad \rho = \frac{G(r)}{gr} (\hat{r} \times \tau)
\]

**Boundary Conditions**

\[
F(0) = \pi, \quad F(\infty) = 0,
\]

\[
G(0) = -2, \quad G(\infty) = 0,
\]

\[
W'(0) = 0, \quad W(\infty) = 0.
\]

**Collective Quantization**

\[
\xi(r) \rightarrow \xi(r, t) = A(t) \xi(r) A^\dagger(t),
\]

\[
V_\mu(r) \rightarrow V_\mu(r, t) = A(t) V_\mu(r) A^\dagger(t),
\]

\[
i\tau \cdot \Omega = A^\dagger(t) \partial_0 A(t).
\]

\[
\rho^0(r, t) = A(t) \frac{2}{g} [\tau \cdot \Omega \xi_1(r) + \hat{r} \cdot \hat{r} \Omega \cdot \hat{r} \xi_2(r)] A^\dagger(t),
\]

\[
\omega^i(r, t) = \frac{\varphi(r)}{r} (\Omega \times \hat{r})^i,
\]

**Boundary Conditions**

\[
\xi_1(0) = \xi_1(\infty) = 0,
\]

\[
\xi_2(0) = \xi_2(\infty) = 0,
\]

\[
\varphi(0) = \varphi(\infty) = 0.
\]

---

**TABLE II.** Skyrmion mass and size calculated in the HLS with the SS and BPS models with \( a = 2 \). The soliton mass \( M_{\text{sol}} \) and the \( \Delta-N \) mass difference \( \Delta_M \) are in units of MeV while \( \sqrt{\langle r^2 \rangle}_W \) and \( \sqrt{\langle r^2 \rangle}_E \) are in units of fm. The column of \( O(p^2) + \omega_\mu B^\mu \) is “the minimal model” of Ref. [20] and that of \( O(p^2) \) corresponds to the model of Ref. [19]. See the text for more details.

<table>
<thead>
<tr>
<th></th>
<th>HLS (_1) (( \pi, \rho, \omega ))</th>
<th>HLS (_1) (( \pi, \rho ))</th>
<th>HLS (_1) (( \pi ))</th>
<th>BPS (( \pi, \rho, \omega ))</th>
<th>BPS (( \pi, \rho ))</th>
<th>BPS (( \pi ))</th>
<th>( O(p^2) + \omega_\mu B^\mu ) [20]</th>
<th>( O(p^2) ) [19]</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_{\text{sol}} )</td>
<td>1184</td>
<td>834</td>
<td>922</td>
<td>1162</td>
<td>577</td>
<td>672</td>
<td>1407</td>
<td>1026</td>
</tr>
<tr>
<td>( \Delta_M )</td>
<td>448</td>
<td>1707</td>
<td>1014</td>
<td>456</td>
<td>4541</td>
<td>2613</td>
<td>259</td>
<td>1131</td>
</tr>
<tr>
<td>( \sqrt{\langle r^2 \rangle}_W )</td>
<td>0.433</td>
<td>0.247</td>
<td>0.309</td>
<td>0.415</td>
<td>0.164</td>
<td>0.225</td>
<td>0.540</td>
<td>0.278</td>
</tr>
<tr>
<td>( \sqrt{\langle r^2 \rangle}_E )</td>
<td>0.608</td>
<td>0.371</td>
<td>0.417</td>
<td>0.598</td>
<td>0.271</td>
<td>0.306</td>
<td>0.725</td>
<td>0.422</td>
</tr>
</tbody>
</table>
The meson reduces the soliton mass from 922 MeV in the HLS($\pi$) to 834 MeV in the HLS($\pi; \rho$) but $\omega$ meson increases the soliton mass to 1184 MeV. This is contrast to the naive expectation that including more vector mesons would decrease the soliton mass.

The $\omega$ meson interacts with the other mesons through the $hWZ$ terms $\Rightarrow$ the importance of the $hWZ$ terms in Skyrme phenomenology.

The role of the vector mesons can also be verified by comparing the soliton wave functions.

All these behaviors can be found in the Skyrmin radii.

In $\Delta M$, the role of the $\rho$ and $\omega$ mesons are the opposite to the case of the soliton mass.

$\Delta M$ increases by the inclusion of the $\rho$ meson from 1014 MeV in the HLS($\pi$) to 1707 MeV in the HLS($\pi; \rho$). $\Rightarrow$ Worsen the situation.

The collective energy at $O(1/N_c)$ is larger than the soliton mass at $O(N_c)$. 

Validity of the collective quantization method?

Inclusion of the $\omega$ meson reduces $\Delta M$ appreciably.

$\xi_{1;2} (r)$ expands in the HLS($\pi; \rho; \omega$) model than in the HLS($\pi; \rho$) model. This results in a larger value for the moment of inertia in the HLS($\pi; \rho; \omega$) model, which leads to a smaller $\Delta M$.

Parameter $a$ dependence of the contributions from each term:

- $O(p^4)$ terms give a negative contribution and the magnitude is much smaller than the $O(p^2)$ terms, $\Rightarrow$ the chiral counting is reasonable.
- Contribution from the $hWZ$ terms is stable as $a$ becomes smaller while the $O(p^2)$ contribution decreases.
- Since the $O(p^2)$ terms and the $O(p^4)$ terms have the opposite $a$ dependence, their sum has very mild $a$ dependence and so does the soliton mass and moment of inertia.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>HLS$_1$($\pi, \rho, \omega$)</th>
<th>HLS$_1$($\pi, \rho$)</th>
<th>HLS$_1$($\pi$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{\text{sol}}$</td>
<td>1184</td>
<td>834</td>
<td>922</td>
</tr>
<tr>
<td>$\Delta M$</td>
<td>448</td>
<td>1707</td>
<td>1014</td>
</tr>
<tr>
<td>$\sqrt{\langle r^2 \rangle_W}$</td>
<td>0.433</td>
<td>0.247</td>
<td>0.309</td>
</tr>
<tr>
<td>$\sqrt{\langle r^2 \rangle_E}$</td>
<td>0.608</td>
<td>0.371</td>
<td>0.417</td>
</tr>
</tbody>
</table>
III. Skyrmion matter from HLS

FCC Skyrmion crystal

Product Ansatz
\[ U(\vec{x}) = AU(\vec{x} - \vec{x}_1)A^\dagger B U(\vec{x}). \]

A) no relative rotation.
B) rotated by an angle \( \pi/2 \) wrt the axis parallel to the line joining the two skyrmions.
C) the angle is \( \pi \) and the rotation axis is perpendicular to the line joining the two skyrmions.

U(x+L/2, y+L/2, z) = τ_z U(x, y, z) τ_z
\[
U(x+L/2, y, z+L/2) = τ_y U(x, y, z) τ_y
\]
\[
U(x, y+L/2, z+L/2) = τ_x U(x, y, z) τ_x
\]
\[
\bar{\phi}_0 = \sum_{abc} \beta_{abc}^t \cos(\frac{\pi a x}{L}) \cos(\frac{\pi b y}{L}) \cos(\frac{\pi c z}{L}),
\]
\[
\bar{\phi}_1 = \sum_{hkl} \alpha_{hkl}^t \sin(\frac{\pi h x}{L}) \cos(\frac{\pi k y}{L}) \cos(\frac{\pi l z}{L}),
\]
\[
\bar{\phi}_2 = \sum_{hkl} \alpha_{hkl}^t \cos(\frac{\pi l x}{L}) \sin(\frac{\pi h y}{L}) \cos(\frac{\pi k z}{L}),
\]
\[
\bar{\phi}_3 = \sum_{hkl} \alpha_{hkl}^t \cos(\frac{\pi k x}{L}) \cos(\frac{\pi l y}{L}) \sin(\frac{\pi h z}{L}),
\]

Symmetries of the FCC skyrmion crystal

<table>
<thead>
<tr>
<th></th>
<th>Reflection (yz-plane)</th>
<th>3-fold axis rotation</th>
<th>4-fold axis (z-axis) rot.</th>
<th>Translation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x, y, z)</td>
<td>→ (-x, y, z)</td>
<td>(y, z, x)</td>
<td>(x, z, -y)</td>
<td>(x + L, y + L, z)</td>
</tr>
<tr>
<td>( U = \sigma + i \bar{\tau} \cdot \vec{\tau} )</td>
<td>→ (\sigma, -\pi_1, \pi_2, \pi_3)</td>
<td>(\sigma, \pi_2, \pi_3, \pi_1)</td>
<td>(\sigma, \pi_1, \pi_3, -\pi_2)</td>
<td>(\sigma, -\pi_1, -\pi_2, \pi_3)</td>
</tr>
<tr>
<td>( \rho_i^t \equiv \epsilon_{aip} \hat{\rho}_p )</td>
<td>→ (-\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3)</td>
<td>(\hat{\rho}_2, \hat{\rho}_3, \hat{\rho}_1)</td>
<td>(\hat{\rho}_1, \hat{\rho}_3, -\hat{\rho}_2)</td>
<td>(-\hat{\rho}_1, -\hat{\rho}_2, \hat{\rho}_3)</td>
</tr>
<tr>
<td>( \omega_0, \chi )</td>
<td>→ \omega_0, \chi</td>
<td>\omega_0, \chi</td>
<td>\omega_0, \chi</td>
<td>\omega_0, \chi</td>
</tr>
</tbody>
</table>
In HLS(π, ρ, ω), \( n_{1/2} \approx n_0 \). Comparing to HLS\(_{min}(π, ρ, ω)\), \( O(p^4) \) terms and the other π-ρ-ω interactions through hWZ terms makes \( n_{1/2} \) larger by a factor 1.7. A noticeable improvement.

This higher \( n_{1/2} \) may come from the fact that the size of the single skyrmion is smaller in the former and that the additional interactions in HLS(π, ρ, ω) weaken the repulsive interactions from the omega.

Supported by the results from HLS(π, ρ), where \( n_{1/2} \approx 6n_0 \). The absence of ω reduces the skyrmion size to almost half compared to that of HLS(π, ρ, ω).

The weak dependence of \( E/B \) on density in HLS(π, ρ) is noticeable. Inclusion of the ρ mesons reduces the soliton mass a lot and almost saturates the Bogololny 's bound.

\( n_{min} > n_{1/2} \) for all three cases. The binding energy \( \sim 150 \) MeV, \( \sim 100 \) MeV and \( \sim 50 \) MeV. As a complete theory of nuclear matter, \( n_{min} \) should represent the nuclear matter ground state. \( n_{min} > n_0 \) and the binding energy at \( n_{min} \) is larger than the empirical value, there is something missing in the present skyrmion crystal description for the EoS of nuclear matter.
The $O(p^4)$ contributes always a attractive interaction.

$(E/B)_{O(p^2)\pi,\rho}$ and $(E/B)_{O(p^2)\omega}$ show distinctly different density dependence in the skyrmion phase and in the half-skyrmion phase. In the single skyrmion phase, $(E/B)_{O(p^2)\pi,\rho}$ dominates over and $(E/B)_{O(p^2)\omega}$, with their even increasing from $\sim 2 - \sim 3$ as the density increases whereas in the half-skyrmion phase, and $(E/B)_{O(p^2)\pi,\rho}$ starts to decrease while $(E/B)_{\omega}$ starts to increase. Then, at some density the repulsive interaction due to the $\omega$ becomes dominant.
IV. Hadron properties with the FCC crystal background

**In-medium modification of meson properties:**

Make substitution:

\[ \xi_{L,R} = \xi_{0,L,R}, \]
\[ \rho^a_\mu = \rho^{a(0)}_\mu, \]
\[ \omega^a_\mu = \omega^{a(0)}_\mu, \]
\[ \tilde{\xi}_{L,R}, \tilde{\rho}^a_\mu, \tilde{\omega}^a_\mu, \]

\[ \mathcal{L}_{(2)} = f_{\pi}^2 \text{Tr}(\hat{a}_{\perp\mu} \hat{a}^\mu_{\perp}) + af_{\pi}^2 \text{Tr}(\hat{a}_{\parallel\mu} \hat{a}^\mu_{\parallel}) \]

\[ \mathcal{L} = \frac{1}{2} \left( 1 - (a - 1) \frac{2}{3} (1 - \langle \sigma^2_{(0)} \rangle) \right) \partial_\mu \pi^a \partial^\mu \pi^a \]

\[ \frac{f_{\pi}^*}{f_{\pi}} = \sqrt{1 - (a - 1) \frac{2}{3} (1 - \langle \sigma^2_{(0)} \rangle)} \]
The density dependence of $f_{\pi}^*/f_{\pi}$ shows the different behavior in the single skyrmion phase and in the half-skyrmion phase.

- In the single skyrmion phase, $f_{\pi}^*/f_{\pi}$ decreases as the baryon number density increases.
- In the half-skyrmion phase, it stays in a non-vanishing value around 0.65.
- Might indicate that, in the half-skyrmion phase, chiral symmetry is broken due to the multi-quark condensate.
In HLS Lagrangian all the parameters are determined by the master formula given in the matter-free space in terms of $f_\pi$ and $m_\rho$.

As the simplest approximation in applying it to dense matter, we shall assume that the in-medium modification of all the parameters involved, can be calculated in terms of the in-medium constants $f_\pi^*$ and $m_\rho^*$.

Now given these starred parameters, we can solve for the single skyrmion solution to obtain the in-medium modification of the baryon mass in dense medium.

To get the leading corrections, it suffices to use the $O(p^2)$ Lagrangian with $a = 2$ as suggested by phenomenology to calculate the two in-medium constants, in which case the most important effect is lodged in the pion-decay constant.

The effects of the $O(p^4)$ terms so calculated can then be straightforwardly incorporated with the starred quantities.
The behaviour of the crystal size dependence of the soliton mass is the same as the behavior of $f_\pi^*$. 

Support by $M_{\text{sol}} \propto f_\pi^*/e = f_\pi^*/g = f_\pi^*/2/m_{\rho}^* \sim N_c$ which follows simply from the scaling of the nucleon mass in the $N_c$ limit as in matter-free space.

The most significant finding of this work: The nucleon mass which decreases with increasing density in the skyrmion phase, stops dropping at $n_{1/2}$ and stays constant in the half-skyrmion phase. In the case of HLS($\pi, \rho, \omega$) $\sim 0.6 M_{\text{sol}}$. 
- The crystal size dependence of the $\Delta$-N mass difference strongly depends on the $\omega$ meson.

- In the calculation including $\omega$ meson, $\Delta M$ decreases up to $n_{1/2}$ beyond which $\Delta M$ becomes nearly constant.

- In the case of HLS($\pi, \rho$), in the low density, $\Delta M$ is unscaling, but near $n_{1/2}$, it scales up and after $n_{1/2}$ $\Delta M$ becomes almost constant.
Skyrmion radius is strongly influenced by the presence of the $\omega$ meson.

In the case of HLS($\pi, \rho$), the skyrmion radius is small and constant at varying density. This is simply because in the absence of $\omega$, the skyrmion is almost point-like and remains stable with the crystal size.
IV. Summary and Discussion

- Baryonic matter properties are studied using the HLS to $O(p^4)$ ($\pi$, $\rho$, $\omega$). We use a general master formula to determine all the LECs self-consistently from a class of hQCD models.

- The hWZ terms in the HLS induced from the CS term in hQCD models are crucial for the inclusion of the omega meson effect in the baryon and baryonic matter properties.

- The results clearly show that both the $\rho$ meson attractive interaction and the $\omega$ meson repulsive interaction affects on the Skyrmion properties.

- Some of our results, such as $n_{\text{min}}$ and binding energy, are deviated from nature. Other meson such as sigma meson effects?
- Other mesons such as sigma (dilaton) meson effects? Necessary attraction for binding from MF!
- Casimir effect (higher loop orders in S-channel in ChPT)? Skyrme model to yield correct nucleon mass.
Principal finding: Simulated on the crystal lattice, the nucleon mass decreases smoothly as density is increased up to $n_{1/2}$ and in the half-skyrmion phase the dropping of the nucleon mass stops and the mass remains constant going toward to the chiral restoration.

In the skyrmion phase with $n < n_{1/2}$, chiral symmetry is broken with the order parameter $<q> \neq 0$. Hadrons are massive apart from the Nambu-Goldstone bosons, the pions.

In the half-skyrmion phase, $<q> = 0$ on the average, but chiral symmetry is not restored since hadrons are still massive and there are pions. There seems to be no obvious order parameter characterizing this state apart from the presence of half-skyrmion structure.

The nucleon mass remains non-vanishing as one approaches the chiral transition point is supported by other approaches.