Discrete Space-Time Symmetries

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Discrete Space-Time Symmetries

- Lorentz Transformations
- Parity
- Charge Conjugation
- Time Reversal
- The CPT Theorem
- Spontaneous Breaking of CP and Cosmology

Lorentz Transformations

- Lorentz transformations

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \, x^\nu$$

preserve the invariance of the space-time interval

$$x^\mu \; x_\mu = \mathbf{x}^2 - c^2t^2 = \mathbf{x'}^2 - c^2t'^2 = \; x'^\mu \; x'_\mu$$

- This constrains the matrices $\Lambda^\mu{}_\nu$ to obey

$$\eta_{\mu\nu} = \Lambda^\lambda{}_\mu \, \eta_{\lambda\kappa} \, \Lambda^\kappa{}_\nu$$

where $\eta_{00} = -1$;$\eta_{ij} = \delta_{ij}$; $\eta_{0i} = \eta_{i0} = 0$

- Pseudo orthogonality of Λ matrices $[\eta = \Lambda^T \eta \, \Lambda]$ allows classification of transformations depending on whether:

$$\det \Lambda = \pm 1 \; ; \; \Lambda^0{}_0 = \pm \, [1 + \Sigma_i \, (\Lambda^i{}_i)^2]^{1/2}$$

- As a result the Lorentz group splits into 4 pieces:

$$L^{\uparrow}_{+}: \quad \det \Lambda = 1 \quad \Lambda^0{}_0 \geq 1$$
$$L^{\uparrow}_{-}: \quad \det \Lambda = -1 \quad \Lambda^0{}_0 \geq 1$$
$$L^{\downarrow}_{+}: \quad \det \Lambda = 1 \quad \Lambda^0{}_0 \leq -1$$
$$L^{\downarrow}_{-}: \quad \det \Lambda = -1 \quad \Lambda^0{}_0 \leq -1$$

- The transformation matrices Λ in $L^{\uparrow}_{+}$ form a subgroup-the proper orthochronous Lorentz group. All other transformations in the Lorentz group can be obtained from $\Lambda \subset L^{\uparrow}_{+}$ by using two discrete transformation Parity $P^{\mu}{}_{\nu} \equiv -\eta_{\mu\nu}$ and Time Reversal $T^{\mu}{}_{\nu} \equiv +\eta_{\mu\nu}$
- Clearly if $\Lambda \subset L^{\uparrow}_{+}$ then $P\Lambda \subset L^{\uparrow}_{-}$; $PT\Lambda \subset L^{\downarrow}_{+}$; and $T\Lambda \subset L^{\downarrow}_{-}$

- Remarkably, nature is invariant under the proper orthochronous Lorentz group $L^{\uparrow}_{+}$ but not the full Lorentz group

\- Parity is violated in the weak interactions

\- Time reversal is violated in K and B decays

- Can understand this on the basis of the Standard Model of the electroweak and strong interactions and of the CPT Theorem [Pauli, Schwinger, Luders, Zumino]
- To understand this, I need to sketch how quantum fields behave under the discrete space-time transformations P and T, as well as under charge conjugation C, which physically corresponds to changing the sign of all charges in the theory

Parity

- The transformation properties of electromagnetic fields under Parity follows directly from classical considerations by looking at the Lorentz force

$$\mathbf{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = e[\mathbf{E} + \mathbf{v} \times \mathbf{B}]$$

Since under Parity $\mathbf{x} \rightarrow -\mathbf{x}$, also $\mathbf{p}, \mathbf{v} \rightarrow -\mathbf{p}, -\mathbf{v}$. Thus it follows that $\mathbf{E}(\mathbf{x},t) \rightarrow -\mathbf{E}(-\mathbf{x},t)$ and $\mathbf{B}(\mathbf{x},t) \rightarrow \mathbf{B}(-\mathbf{x},t)$

- For the gauge potential A^μ, more formally, the Parity transformation is induced by a unitary operator U(P) which gives:

$$U(P)\, A^\mu(\mathbf{x},t)\, U(P)^{-1} = \eta[\mu]\, A^\mu(-\mathbf{x},t)\ ;\ (\eta[0]=1;\ \eta[i]=-1)$$

- Spin zero scalar (S) and pseudoscalar (P) fields under Parity transform as follows:

$$U(P)\,S(\mathbf{x},t)\,U(P)^{-1} = S(-\mathbf{x},t)$$

$$U(P)\,P(\mathbf{x},t)\,U(P)^{-1} = -P(-\mathbf{x},t)$$

- For spin ½ Dirac fields ψ one can deduce the Parity transformation of the fields from the requirement that the Dirac equation

$$[-i\,\gamma^{\mu}\partial_{\mu} + m]\,\psi = 0$$

be left invariant under the replacement of $\mathbf{x} \to -\mathbf{x}$ and $\psi(\mathbf{x},t) \to \psi(-\mathbf{x},t)$. This is achieved if under Parity

$$U(P)\,\psi(\mathbf{x},t)\,U(P)^{-1} = \eta_P\,\gamma^0\psi(-\mathbf{x},t) \quad \text{with} \quad |\eta_P|^2=1$$

- Using that $\{\gamma^{\mu}, \gamma^{\nu}\} = -2\,\eta^{\mu\nu}$ and that $\gamma^{0\dagger} = \gamma^0$ while $\gamma^{i\dagger} = -\gamma^i$ and defining a matrix $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, which obeys $\{\gamma_5, \gamma_{\mu}\} = 0$, $\gamma_5^{\dagger} = \gamma_5$ and $\gamma_5^{\,2} = 1$, one can easily deduce the Parity properties of fermion bilinears

- Consider, for example, the bilinear

$$\bar{\psi}(\mathbf{x},t)\,\psi(\mathbf{x},t) = \psi(\mathbf{x},t)^{\dagger}\,\gamma^0\,\psi(\mathbf{x},t)$$

Then

$$U(P)\;\bar{\psi}(\mathbf{x},t)\psi(\mathbf{x},t)\;U(P)^{-1} = U(P)\;\psi(\mathbf{x},t)^{\dagger}\;U(P)^{-1}\gamma^0$$

$$\otimes\;U(P)\;\psi(\mathbf{x},t)\;U(P)^{-1}$$

$$= \psi(-\mathbf{x},t)^{\dagger}\,\eta_P{}^{*}\,\gamma^{0\,\dagger}\gamma^0$$

$$\otimes\;\eta_P\,\gamma^0\psi(-\mathbf{x},t)$$

Since $\gamma^{0\,\dagger}\gamma^0\gamma^0 = \gamma^0$ and $|\eta_P|^2=1$ it follows that

$$U(P)\;\bar{\psi}(\mathbf{x},t)\psi(\mathbf{x},t)\;U(P)^{-1} = \bar{\psi}(-\mathbf{x},t)\psi(-\mathbf{x},t)$$

- One finds in similar fashion that the transformation properties for scalar, pseudoscalar, vector and axial vector fermion bilinears are:

$$U(P)\ \bar{\psi}(\mathbf{x},t)\psi(\mathbf{x},t)\ U(P)^{-1} = \bar{\psi}(-\mathbf{x},t)\psi(-\mathbf{x},t)$$

$$U(P)\ \bar{\psi}(\mathbf{x},t)\gamma_5\psi(\mathbf{x},t)\ U(P)^{-1} = -\ \bar{\psi}(-\mathbf{x},t)\gamma_5\psi(-\mathbf{x},t)$$

$$U(P)\ \bar{\psi}(\mathbf{x},t)\gamma^\mu\psi(\mathbf{x},t)\ U(P)^{-1} = \eta[\mu]\ \bar{\psi}(-\mathbf{x},t)\gamma^\mu\psi(-\mathbf{x},t)$$

$$U(P)\ \bar{\psi}(\mathbf{x},t)\gamma^\mu\gamma_5\psi(\mathbf{x},t)\ U(P)^{-1} = -\eta[\mu]\ \bar{\psi}(-\mathbf{x},t)\gamma^\mu\gamma_5\psi(-\mathbf{x},t)$$

- From the above, one sees immediately that the electromagnetic interactions are invariant under Parity:

$$W_{em} = \int d^4x\ e\ A^\mu(x)\ \bar{\psi}(x)\gamma_\mu\psi(x) \xrightarrow{\text{Parity}} W_{em}$$

- It is interesting to consider how Parity transforms fields of a given chirality:

$$\psi_L = \frac{1}{2}(1-\gamma_5)\,\psi \; ; \; \psi_R = \frac{1}{2}(1+\gamma_5)\,\psi$$

- Since $\{\gamma_\mu, \gamma_5\} = 0$, is easy to see that

$$U(P)\,\psi_L(\mathbf{x},t)\,U(P)^{-1} = \eta_P\,\gamma^0\psi_R(-\mathbf{x},t)$$

$$U(P)\,\psi_R(\mathbf{x},t)\,U(P)^{-1} = \eta_P\,\gamma^0\psi_L(-\mathbf{x},t)$$

- Thus, chiral symmetric interactions are Parity conserving and chiral asymmetric interactions violate Parity:

- In QCD, based on an SU(3) gauge theory, the quark fields 3_L and 3_R interact in the same way. Hence, Parity is conserved in the strong interactions
- In the SU(2) x U(1) electroweak theory, under SU(2) $\psi_L \sim 2$ while $\psi_R \sim 1$. Hence, the weak interactions violate Parity

Charge Conjugation

- Physically Charge Conjugation C is associated with changing the sign of all charges. Hence for the electromagnetic potential one has:

$$U(C)\, A^{\mu}(x)\, U(C)^{-1} = -\, A^{\mu}(x)$$

- For Dirac particles, since under C one wants to transform particles into antiparticles, it corresponds, essentially, to Hermitian conjugation

$$U(C)\, \psi(x)\, U(C)^{-1} = \eta_C\, C\, \psi^{\dagger}(x) \quad \text{with} \quad |\eta_C|^2 = 1$$

The matrix C can be deduced from requiring that the Dirac equation be invariant under C, which gives

$$C\, \gamma^{\mu *} C^{-1} = -\, \gamma^{\mu}$$

Actually, C depends on the form of the γ-matrices used:

Majorana repr: $\gamma^{\mu *} = -\, \gamma^{\mu} \Rightarrow C = 1$

Dirac repr: $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \Rightarrow C = \gamma^2$

- It is straightforward to compute how fermion bilinears behave under charge conjugation. For example:

$$U(C)\ \bar{\psi}(x)\psi(x)\ U(C)^{-1} = U(C)\ \psi^{\dagger}{}_{\alpha}(x)\ \gamma^{0}{}_{\alpha\beta}\ \psi_{\beta}(x)\ U(C)^{-1}$$

{Majorana repr. $C=1$} $= \psi_{\alpha}(x)\ \gamma^{0}{}_{\alpha\beta}\ \psi^{\dagger}{}_{\beta}(x)$

{Fermions anti com.} $= -\ \psi^{\dagger}{}_{\beta}(x)\ \gamma^{0}{}_{\alpha\beta}\ \psi_{\alpha}(x)$

$= -\ \psi^{\dagger}{}_{\beta}(x)\ \gamma^{0T}{}_{\beta\alpha}\ \psi_{\alpha}(x)$

{Majorana $\gamma^{0T} = -\ \gamma^{0}$} $= \bar{\psi}(x)\psi(x)$

- Similar calculations lead to the results:

$$U(C)\ \bar{\psi}(x)\ \gamma_5\ \psi(x)\ U(C)^{-1} = \bar{\psi}(x)\ \gamma_5\ \psi(x)$$

$$U(C)\ \bar{\psi}(x)\ \gamma^{\mu}\ \psi(x)\ U(C)^{-1} = -\ \bar{\psi}(x)\ \gamma^{\mu}\ \psi(x)$$

$$U(C)\ \bar{\psi}(x)\ \gamma^{\mu}\gamma_5\ \psi(x)\ U(C)^{-1} = \bar{\psi}(x)\ \gamma^{\mu}\gamma_5\ \psi(x)$$

- An immediate consequence of these results is that electromagnetic interactions are invariant under charge conjugation

$$W_{em} = \int d^4x\, e\, A^{\mu}(x)\, \bar{\psi}(x)\gamma_{\mu}\psi(x) \xrightarrow{\text{Charge Conj.}} W_{em}$$

- Besides QED also QCD is C-invariant, since it involves only vector interactions. However, the SU(3) currents $J^{\mu}{}_a(x) = \bar{q}(x)\gamma^{\mu}\lambda_a q(x)$ do not transform as simply under C as $J^{\mu}{}_{em}(x) = \bar{\psi}(x)\gamma^{\mu}\psi(x)$ does
- One has:

$$U(C)\; \bar{q}(x)\gamma^{\mu}\lambda_a q(x)\; U(C)^{-1} = -\; \bar{q}(x)\,\gamma^{\mu}\lambda^T{}_a q(x)$$

- Now $\lambda^T{}_a = -\lambda_a$ for a=2,5,7, while $\lambda^T{}_a = \lambda_a$ for a=1,3,4,6,8. Thus for QCD to be C-invariant we require that under C:

$$A_a{}^{\mu}(x) \rightarrow A_a{}^{\mu}(x) \;\; a=2,5,7 \;\; ; \; A_a{}^{\mu}(x) \rightarrow -A_a{}^{\mu}(x) \;\; a=1,3,4,6,8$$

or

$$A_a{}^{\mu}(x) \rightarrow -\eta[a]\, A_a{}^{\mu}(x)$$

with $\eta[a]=1$ for a=1,3,4,6,8 and $\eta[a]= -1$ for a=2,5,7

- One can check that these transformation properties for $A_a^\mu(x)$ are precisely what is needed to have the field strengths $G_a^{\mu\nu}(x)$ have well defined transformation properties under C
- Since the only non-vanishing structure constants f_{abc} are for abc=123; 147; 156; 246; 257; 345; 367; 458; 678, it is easy to see that

$$G_a^{\mu\nu}(x) = \partial^\mu A_a^\nu(x) - \partial^\nu A_a^\mu(x) + g_3 f_{abc} A_b^\mu(x) A^\nu_c(x)$$

indeed transforms under C just as $A_a^\mu(x)$ does:

$$G_a^{\mu\nu}(x) \to G_a^{\mu\nu}(x) \; a=2,5,7 \; ; \; G_a^{\mu\nu}(x) \to -G_a^{\mu\nu}(x) \; a=1,3,4,6,8$$

Or

$$G_a^{\mu\nu}(x) \to -\eta[a] \, G_a^{\mu\nu}(x)$$

- This insures that under charge conjugation

$$W_{QCD} = \int d^4x \, \{ \; \bar{q}(x)[i\gamma^\mu D_\mu - m_q] \, q(x) - 1/4 \, G_a^{\mu\nu}(x) \, G_{a\mu\nu}(x)\} \to W_{QCD}$$

- The situation is different for the weak interactions since they involve both vector and axial currents in the action
- For instance, for the SU(2) interactions one has

$$W^{int} = \int d^4x \, g_2 \, J^{\mu}{}_i(x) \, W_{\mu\, i}(x)$$

where , say, for the 1st generation of leptons:

$$J^{\mu}{}_i(x) = (\, \overline{\nu}_e(x) \;\; \overline{e}(x))_L \, \gamma^{\mu} \, \tau_i \begin{pmatrix} \nu_e(x) \\ e(x) \end{pmatrix}_L$$

$$= 1/2 \, (\, \overline{\nu}_e(x) \;\; \overline{e}(x)) \, \gamma^{\mu}(1 - \gamma_5) \, \tau_i \begin{pmatrix} \nu_e(x) \\ e(x) \end{pmatrix}$$

- These currents transform differently under C in their vector and axial vector parts, as well as in their 1, 2, or 3 components.
- Although one can compensate for the SU(2) dependence of $J^{\mu}{}_i(x)$ with appropriate C-transformation properties for the $W_{\mu\, i}(x)$ fields, the presence of both vector and axial currents in $J^{\mu}{}_i(x)$ violates charge conjugation invariance

- A straightforward calculation gives:

$$U(C)J^{\mu}{}_{1,3}(x)U(C)^{-1} = -1/2(\ \bar{\nu}_e(x)\ \ \bar{e}(x))\gamma^{\mu}(1+\gamma_5)\ \tau_{1,3}\begin{pmatrix}\nu_e(x)\\ e(x)\end{pmatrix}$$

$$U(C)J^{\mu}{}_{2}(x)U(C)^{-1} = 1/2(\ \bar{\nu}_e(x)\ \ \bar{e}(x))\gamma^{\mu}(1+\gamma_5)\ \tau_{2}\begin{pmatrix}\nu_e(x)\\ e(x)\end{pmatrix}$$

- The difference in the behavior in the 1,3 and 2 components can be absorbed by postulating that

$$U(C)\ W^{\mu}{}_{i}(x)\ U(C)^{-1} = -\eta[i]\ W^{\mu}{}_{i}(x)$$

where $\eta[1,3] = 1$; $\eta[2] = -1$

- This is as one might expect since it implies that, under C, the charged W fields transform as:

$$W^{\mu}{}_{+/-}(x) = i/\sqrt{2}[W^{\mu}{}_{1}(x) -/+ iW^{\mu}{}_{2}(x)] \rightarrow -\ W^{\mu}{}_{-/+}(x)$$

- However, even so, the simultaneous presence of both vector and axial pieces in $J^{\mu}{}_{i}(x)$ renders W^{int} not invariant under C
- Writing, in an obvious notation,

$$J^{\mu}{}_{i}(x) = V^{\mu}{}_{i}(x) - A^{\mu}{}_{i}(x)$$

then one sees that under C

$$W^{int} = \int d^4x \, g_2 [V^{\mu}{}_{i}(x) - A^{\mu}{}_{i}(x)] W_{\mu\, i}(x)$$
$$\rightarrow \int d^4x \, g_2 [V^{\mu}{}_{i}(x) + A^{\mu}{}_{i}(x)] W_{\mu\, i}(x)$$

Thus C is violated by the weak interactions

- I remark that the presence of the axial currents is also what causes Parity to be violated. However, note that under the combined operation of C and P

$$W^{int} \rightarrow W^{int}$$

at least for the limited sector we have explored [more later]

Time Reversal

- Classically, T-invariance corresponds to having as permitted motions both those going forward in time [$\rightarrow$]as well as backwards in time [$\leftarrow$].
- Of course, under T, dynamical variables change appropriately: $t \rightarrow -t$; $\mathbf{x} \rightarrow \mathbf{x}$; $\mathbf{p} \rightarrow -\mathbf{p}$; $\mathbf{F} \rightarrow \mathbf{F}$
- Quantum mechanically the interchange of initial and final states is implemented by having U(T) be an anti-unitary operator : [Wigner]

$$U(T) = V(T)\,K$$

where $V^{\dagger}(T) = V^{-1}(T)$ and $K \equiv$ complex conjugation

- The need for complex conjugation is seen directly from the Schroedinger equation

- Taking the complex conjugate of the Schroedinger equation

$$i\frac{\partial}{\partial t}\Psi(\mathbf{x}, t) = H\ \Psi(\mathbf{x}, t)$$

and letting $t \rightarrow -t$ gives the equation

$$i\frac{\partial}{\partial t}\Psi^*(\mathbf{x}, -t) = H^*\ \Psi^*(\mathbf{x}, -t)$$

- So, provided the Hamiltonian is real ($H^* = H$), if $\Psi(\mathbf{x}, t)$ is a solution of the Schroedinger equation, so is $\Psi^*(\mathbf{x}, -t)$
- For T-invariance asking for the reality of H needs slight modification if spin is involved. More correctly, what is needed is that

$$V(T)\ H^*\ V(T)^{-1} = H$$

- For example, for spin orbit coupling $H_{so} = \gamma\ \boldsymbol{\sigma}.\mathbf{L}$ and since, under T, $\mathbf{L} \rightarrow -\mathbf{L}$ one has $H^*_{so} = -\gamma\boldsymbol{\sigma}^*.\mathbf{L}$. This can be returned to its original form using $V(T) = \sigma_2$, since $\sigma_2\ \boldsymbol{\sigma}^*\sigma_2 = -\ \boldsymbol{\sigma}$.
- Thus we learn that under T not only $\mathbf{L} \rightarrow -\mathbf{L}$, but also effectively $\boldsymbol{\sigma} \rightarrow -\boldsymbol{\sigma}$. Time reversal, reverses all spins!

- Association of complex conjugation with time reversal interchanges incoming and outgoing states:

$$\langle U(T)\Phi | U(T)\Psi\rangle = \langle \Psi | \Phi \rangle$$

- Thus if T is a good symmetry, one relates processes to their time-reversed process, e.g. $AB \rightarrow CD$ to $CD \rightarrow AB$
- In terms of S-matrix elements, if T is a good symmetry, then

$$S_{fi} = {}_{out}\langle f \mid i\rangle_{in} = {}_{in}\langle U(T)i | U(T)f\rangle_{out} = {}_{out}\langle i_T \mid f_T\rangle_{in}$$

where the last steps follows if T is a good symmetry. In this case then

$$|U(T)\, f\rangle_{out} = |f_T\rangle_{in}$$

where if $f = \{\mathbf{p}_C, \mathbf{p}_D\}$ then $f_T = \{-\mathbf{p}_C, -\mathbf{p}_D\}$

- In field theory the action of T on electromagnetic fields can be gleaned from the behavior of the Lorentz force:

$$\mathbf{F} = e[\mathbf{E} + \mathbf{v} \times \mathbf{B}] \rightarrow \mathbf{F} \Rightarrow \mathbf{E}(\mathbf{x},t) \rightarrow \mathbf{E}(\mathbf{x},t);\ \mathbf{B}(\mathbf{x},t) \rightarrow -\mathbf{B}(\mathbf{x},t)$$

- It follows therefore that:

$$U(T)\, A^{\mu}(\mathbf{x},t)\, U(T)^{-1} = \eta[\mu]\, A^{\mu}(\mathbf{x},-t)\ ;\ (\eta[0]=1;\ \eta[i]=-1)$$

- For Dirac fields one can deduce the field transformation properties under time reversal by again asking that the action of U(T) on the Dirac equation produces another solution for this equation
- Writing

$$U(T)\, \psi(\mathbf{x},t)\, U(T)^{-1} = \eta_T T\, \psi(\mathbf{x},t) \quad \text{with } |\eta_T|^2 = 1$$

and remembering that U(T) complex conjugates all c-numbers, one finds that the matrix T must obey:

$$T\, \gamma^{0*}\, T^{-1} = \gamma^0\ ;\ T\, \gamma^{i*}\, T^{-1} = -\gamma^i$$

Again the form of the matrix T depends on the representation of the γ -matrices used. In the convenient Majorana representation:

$$T = \gamma^0\, \gamma_5$$

- It is straightforward to compute how the various fermion bilinears transform under T. One finds:

$$U(T)\ \bar{\psi}(\mathbf{x},t)\psi(\mathbf{x},t)\ U(T)^{-1} = \bar{\psi}(\mathbf{x},-t)\psi(\mathbf{x},-t)$$

$$U(T)\ \bar{\psi}(\mathbf{x},t)\gamma_5\psi(\mathbf{x},t)\ U(T)^{-1} = \bar{\psi}(\mathbf{x},-t)\gamma_5\psi(\mathbf{x},-t)$$

$$U(T)\ \bar{\psi}(\mathbf{x},t)\gamma^\mu\psi(\mathbf{x},t)\ U(T)^{-1} = \eta[\mu]\ \bar{\psi}(\mathbf{x},-t)\gamma^\mu\psi(\mathbf{x},-t)$$

$$U(T)\ \bar{\psi}(\mathbf{x},t)\gamma^\mu\gamma_5\psi(\mathbf{x},t)\ U(T)^{-1} = \eta[\mu]\ \bar{\psi}(\mathbf{x},-t)\gamma^\mu\gamma_5\psi(\mathbf{x},-t)$$

- Note that, in contrast to C, T-transformations affect vector and axial currents the same way

- It follows immediately from the above, and the fact that the electric charge e is real, that the electromagnetic interactions are conserved under T

$$W_{em} = \int d^4x\ e\ A^\mu(x)\ \bar{\psi}(x)\gamma_\mu\psi(x) \xrightarrow{\text{time reversal}} W_{em}$$

- It is also easy to check that the gauge interactions in both QCD and in the electroweak theory also conserve T, provided one properly defines how the gauge fields transform.

- Since for:

- SU(3): $\lambda^*_a = -\lambda_a$ for a=2,5,7; $\lambda^*_a = \lambda_a$ for a=1,3,4,6,8
- SU(2): $\tau^*_1 = \tau_1$; $\tau^*_2 = -\tau_2$; $\tau^*_3 = \tau_3$

  it is easy to check that the desired transformation properties of the gauge fields under T are:

$$U(T)\, A^{\mu}_a(\mathbf{x},t)\, U(T)^{-1} = \eta[\mu]\,\eta[a]\, A^{\mu}_a(\mathbf{x},-t) \qquad \text{QCD}$$

$$U(T)\, W^{\mu}_i(\mathbf{x},t)\, U(T)^{-1} = \eta[\mu]\,\eta[i]\, W^{\mu}_i(\mathbf{x},-t) \qquad \text{SU(2)}$$

$$U(T)\, Y^{\mu}(\mathbf{x},t)\, U(T)^{-1} = \eta[\mu]\, Y^{\mu}(\mathbf{x},-t) \qquad \text{U(1)}$$

- Since the gauge couplings g_1, g_2, g_3 are real, it follows that under Time Reversal T

$$W_{\text{Gauge int}}[\text{QCD, SU(2)xU(1)}] \rightarrow W_{\text{Gauge int}}[\text{QCD, SU(2)xU(1)}]$$

- However, T-violation can arise in the electroweak theory in the interactions involving the Higgs field, since these couplings can be complex

- Let us examine the simplest example involving just one complex Higgs doublet field Φ:

$$\Phi = \begin{pmatrix} \phi^0 \\ \phi^- \end{pmatrix}$$

- The Higgs self-interactions which cause the breakdown of $SU(2) \times U(1) \rightarrow U_{em}(1)$ are real, since the Higgs potential must be Hermitian:

$$V = \lambda[\Phi^\dagger \Phi - v^2/2]^2 ; V = V^\dagger \Rightarrow \lambda, v^2 \text{ real}$$

- However, the Yukawa interactions detailing the coupling of the Higgs field with the fermions in the theory can have complex coefficients
- For example, for the quark sector, one has:

$$L_{Yukawa} = -\Gamma^u_{ij} \left(\bar{u}_L \quad \bar{d}_L \right)_i \Phi \, u_{Rj} - \Gamma^d_{ij} \left(\bar{u}_L \quad \bar{d}_L \right)_i \Phi' \, d_{Rj} + \text{h.c.}$$

where $\Phi' = i\sigma_2 \Phi^*$ and, in general, the couplings Γ^u_{ij} and Γ^d_{ij} are complex numbers

- After the $SU(2) \times U(1) \rightarrow U_{em}(1)$ breakdown, effectively, the Higgs field gets replaced by:

$$\Phi = \begin{pmatrix} \phi^0 \\ \phi^- \end{pmatrix} \rightarrow 1/\sqrt{2} \begin{pmatrix} v + H \\ 0 \end{pmatrix}$$

where H is the physical Higgs field

- The resulting mass matrices for the quarks

$$M^u_{ij} = v/\sqrt{2}\Gamma^u_{ij} \ ; \ M^d_{ij} = v/\sqrt{2}\Gamma^d_{ij}$$

can be diagonalized by a bi-unitary transformation

$$U^u_L M^u U^u_R = M^u_{diag} \ ; \ U^d_L M^d U^d_R = M^d_{diag}$$

reducing the Yukawa interactions to:

$$L_{Yukawa} \rightarrow \Sigma_i \, m_{q_i} \, \bar{q}q \, [1 + H/v]$$

- This is clearly T-invariant provided that under T

$$H(\mathbf{x},t) \rightarrow H(\mathbf{x},-t)$$

- However, the unitary transformation on quarks to diagonalize their mass matrices alters the form of the charged current weak interactions
- Before this transformation one had:

$$L_{CC} = \frac{g_2}{2\sqrt{2}} \{W_+^{\mu} J_{-\mu} + W_-^{\mu} J_{+\mu}\}$$

where

$$J_{-\mu} = (\bar{u}_1 \quad \bar{u}_2 \quad \bar{u}_3)\gamma_\mu (1-\gamma_5)\, 1 \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix};\quad J_{+\mu} = J^{\dagger}_{-\mu}$$

- After the transformation to the physical quark states the currents now involve the Cabibbo Kobayashi Maskawa unitary matrix V_{CKM} given by :

$$V_{CKM} = U^{\dagger u}{}_L\, U^d{}_L$$

and one has

$$J_{-\mu} = (\bar{u} \quad \bar{c} \quad \bar{t})\gamma_\mu (1-\gamma_5)\, V_{CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix} ; \quad J_{+\mu} = J^{\dagger}_{-\mu}$$

- Because under T the matrix V_{CKM} gets complex conjugated

$$U(T)\, J_{-\mu}\, U(T)^{-1} = \eta[\mu]\, (\bar{u} \quad \bar{c} \quad \bar{t})\gamma_\mu (1-\gamma_5)\, V^*_{CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

while,

$$U(T)\, W_+^{\mu}\, U(T)^{-1} = \eta[\mu]\, W_+^{\mu}$$

the charged current interactions written in this new physical basis violate Time reversal. Under T:

$$U(T)\, W_+^{\mu} J_{-\mu}\, U(T)^{-1} \neq W_+^{\mu} J_{-\mu}$$

- For the case of 3 generations one can show that V_{CKM} has only one phase δ. Thus, T-violation in the Standard Model can be ascribed to the presence of this phase δ

The CPT Theorem

- If nature is described by a local, Lorentz invariant field theory, where there is the usual connection between spin and statistics, then one can prove a deep result, the CPT theorem.
- This theorem states that if the above conditions hold then under CPT transformations the action of the theory is invariant [Pauli, Schwinger, Luders, Zumino]

$$W \xrightarrow{CPT} W$$

- Proof of this theorem can be gleaned from our discussion of the separate P, C, and T transformation properties of quantum fields

- Let us look at QED, as a warm-up. Under the combined C, P, and T transformations one has:

$$A^{\mu}(\mathbf{x},t) \xrightarrow{CPT} [-1]\ \eta[\mu]\ \eta[\mu]\ A^{\mu}(-\mathbf{x},-t)$$

$$\bar{\psi}(\mathbf{x},t)\gamma^{\mu}\psi(\mathbf{x},t) \xrightarrow{CPT} [-1]\ \eta[\mu]\ \eta[\mu]\ \ \bar{\psi}(-\mathbf{x},-t)\gamma^{\mu}\psi(-\mathbf{x},-t)$$

- Thus, obviously, since

$$W_{int}^{QED} = \int d^4x\ e\ A^{\mu}(x)\ \ \bar{\psi}(x)\gamma_{\mu}\psi(x)$$

is separately invariant under C, P, and T transformations, then also

$$W_{int}^{QED} \xrightarrow{CPT} W_{int}^{QED}$$

- However, CPT invariance also holds when there is violation of the separate symmetries

- As an example, consider neutral current interactions in the electroweak theory. These interactions violate both P and C. However, both T and CPT are conserved
- The action for neutral current interactions is:

$$W_{int.}^{NC} = \frac{e}{2\cos\theta_W \sin\theta_W} \int d^4x \, J^{\mu}_{NC} Z_{\mu}$$

where

$$J^{\mu}_{NC} = 2\,\{J^{\mu}_{3} - \sin^2\theta_W J^{\mu}_{em}\} = V^{\mu} + A^{\mu}$$

is the sum of vector and axial currents

- It is easy to see that Parity is violated in $W_{int.}^{NC}$ since:

$$Z_{\mu} \xrightarrow{P} \eta[\mu]\, Z_{\mu}\,;\; V^{\mu} \xrightarrow{P} \eta[\mu]\, V^{\mu}\,;\; A^{\mu} \xrightarrow{P} -\eta[\mu] A^{\mu}$$

- Also Charge Conjugation is violated since:

$$Z_{\mu} \xrightarrow{C} -Z_{\mu}\,;\; V^{\mu} \xrightarrow{C} -V^{\mu}\,;\; A^{\mu} \xrightarrow{C} A^{\mu}$$

- However, T is conserved in $W_{int.}^{NC}$ since:

$$Z_\mu \xrightarrow{T} \eta[\mu]\, Z_\mu \,;\, V^\mu \xrightarrow{T} \eta[\mu]\, V^\mu \,;\, A^\mu \xrightarrow{T} \eta[\mu] A^\mu$$

- And so is CPT, since

$$Z_\mu \xrightarrow{CPT} -Z_\mu \,;\, V^\mu \xrightarrow{CPT} -V^\mu \,;\, A^\mu \xrightarrow{CPT} -A^\mu$$

- Note also that, up to an irrelevant sign, T and CP are equivalent since:

$$Z_\mu \xrightarrow{CP} -\eta[\mu]\, Z_\mu \,;\, V^\mu \xrightarrow{CP} -\eta[\mu]\, V^\mu \,;\, A^\mu \xrightarrow{CP} -\eta[\mu] A^\mu$$

Thus CP is conserved in $W_{int.}^{NC}$

- This equivalence holds also when CP and T are each violated. Hence the combined CPT transformation is an invariance of the action, as required by the CPT theorem

- Let us check this last point by looking at the T-violating CKM interactions. For simplicity, let us just consider the ub piece, where in the standard parameterization $[V_{CKM}]_{ub} = \sin\theta_{13}\, e^{-i\delta}$. Then

$$W_{ub} = \frac{e \sin\theta_{13}}{2\sqrt{2}\,\sin\theta_w} \int d^4x \, \{e^{-i\delta}\, W_+^{\mu} \; \bar{u}\, \gamma_\mu (1-\gamma_5)\, b + \text{h.c.}\}$$

- Because under T:

$$W_+^{\mu} \xrightarrow{T} \eta[\mu]\, W_+^{\mu}$$

and

$$\bar{u}\, \gamma_\mu (1-\gamma_5)\, b \xrightarrow{T} \eta[\mu] \;\; \bar{u}\, \gamma_\mu (1-\gamma_5)\, b$$

but all c-numbers get complex conjugated:

$$W_{ub} \xrightarrow{T} \frac{e \sin\theta_{13}}{2\sqrt{2}\,\sin\theta_w} \int d^4x \, \{e^{+i\delta}\, W_+^{\mu} \; \bar{u}\, \gamma_\mu (1-\gamma_5)\, b + \text{h.c.}\}$$

So, indeed, as we argued earlier, T is violated

- The behavior under CP is individually different, since particles are transformed into anti-particles. However, the net effect is the same as T
- One has:

$$W_+^{\mu} \xrightarrow{CP} -\eta[\mu]\, W_-^{\mu}$$

$$\bar{u}\, \gamma_{\mu}\,(1-\gamma_5)\, b \xrightarrow{CP} -\eta[\mu]\, \bar{b}\, \gamma_{\mu}\,(1-\gamma_5)\, u$$

- Thus, under CP:

$$W_{ub} \xrightarrow{CP} \frac{e \sin\theta_{13}}{2\sqrt{2}\,\sin\theta_W} \int d^4x\, \{e^{-i\delta}\, W_-^{\mu}\, \bar{b}\, \gamma_{\mu}\,(1-\gamma_5)\, u + \text{h.c.}\}$$

$$= \frac{e \sin\theta_{13}}{2\sqrt{2}\,\sin\theta_W} \int d^4x\, \{e^{+i\delta}\, W_+^{\mu}\, \bar{u}\, \gamma_{\mu}\,(1-\gamma_5)\, b + \text{h.c.}\}$$

which is precisely the result we obtained when we did a T-transformation on W_{ub}

- It follows therefore that

$$W_{ub} \xrightarrow{CPT} W_{ub}$$

- More generally, the CPT theorem holds as the result of the Hermiticity of the Lagrangian and the role that T and CP play on the operators in the Lagrangian
- Due to the Hermiticity of the Lagrangian, the most general term has the structure:

$$L = a\,O(x) + a^*\,O^\dagger(x)$$

- Now, under T:

$$O(\mathbf{x}, t) \xrightarrow{T} O(\mathbf{x}, -t)\ ;\ a \xrightarrow{T} a^*$$

while under CP:

$$O(\mathbf{x}, t) \xrightarrow{CP} O^\dagger(-\mathbf{x}, t)\ ;\ a \xrightarrow{CP} a$$

- Hence

$$W = \int d^4x\, L = \int d^4x\, \{a O(x) + a^* O^\dagger(x)\} \xrightarrow{CPT} W$$

which is the CPT theorem

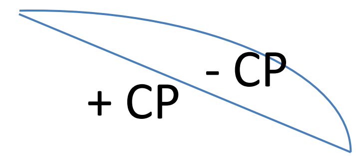
Spontaneous Breaking of CP and Cosmology

- If there are no scalar fields in the theory, the Lagrangian of a theory of just fermions and gauge fields in general conserves CP and T (ignoring θ-terms)
- However, CP and T can be broken spontaneously through the formation of complex fermion condensates

$$< \bar{\psi}(x)\psi(x) > = \Lambda^3 e^{i\delta}$$

- Spontaneous CP-violation however has cosmological consequences, because CP domains, separated by walls, form in the Universe [Kobzarev Okun Zeldovich]

- In Universe find domains of different CP



- These domains are separate by walls with a surface energy density $\sigma \sim \Lambda^3$
- The energy density in these walls dissipates very slowly as the Universe cools, with

$$\rho_{Wall} = \sigma T$$

- As the temperature of the Universe gets below $T \sim \Lambda$ the energy density in the walls begins to dominate and eventually overcloses the Universe
- To avoid this problem one has to assume that the scale of spontaneous CP-violation Λ is above the temperature scale of inflation $\Lambda > T_{inflation}$, so that we effective live in one inflated domain