Relativistic BCS-BEC Crossover in Quark Matter

Pengfei Zhuang

Physics Department, Tsinghua University, Beijing 100084

1) Introduction
2) Mean Field
3) Fluctuations
4) Nuclear Matter and Quark Matter
5) Conclusions
In BCS, $T_c$ is determined by thermal excitation of fermions, in BEC, $T_c$ is controlled by thermal excitation of collective modes.
Introduction: BCS-BEC in Quark Matter

QCD phase diagram

There may exist BCS-BEC crossovers in quark matter!

New phenomena in BCS-BEC crossover of QCD:
- Relativistic systems,
- Anti-fermion contribution,
- Rich inner structure (color, flavor),
- Medium dependent mass, ……
*) non-relativistic mean field theory at $T=0$ (Leggett, 1980)

*) non-relativistic theory at $T \neq 0$ (Nozieres and Schmitt-Rink, 1985)

extension to relativistic system (Nishida and Abuki (2006, 2007))

$$\Omega_{fl} = \int \frac{d^4 q}{(2\pi)^4} \ln \left[ \frac{1}{G} - \chi(q) \right], \quad \chi = \bigcirc \sim G_0 G_0$$

*) non-relativistic $G_0G$ scheme (Chen, Levin et al., 1998, 2000, 2005)

asymmetric pair susceptibility

$$\chi = \bigcirc \sim G_0 G$$

extension to relativistic $G_0G$ scheme (He, Jin, PZ, 2006, 2007)

*) bose-fermion model (Friedderg, Lee, 1989, 1990)

extension to relativistic systems (Deng, Wang, 2007)

*) Kitazawa, Rischke, Shovkovy, 2007, NJL+phase diagram

Brauner, 2008, collective excitations ……
Non-relativistic Mean Field at $T=0$

**A.J. Leggett, 1980**

*universality behavior*

**BCS limit**

\[ \eta = \frac{1}{k_F a_s} \rightarrow -\infty, \quad \tilde{\Delta} = \frac{8}{\epsilon^2} e^{2\eta/\pi}, \quad \hat{\mu} = 1 \]

**BEC limit**

\[ \eta \rightarrow \infty, \quad \tilde{\Delta} = \sqrt{\frac{16\eta}{3\pi}}, \quad \hat{\mu} = -\eta^2 \]

\[ \mu = -\frac{\epsilon_b}{2}, \quad \epsilon_b = \frac{1}{ma_s^2} \]

\[ n(p) = \frac{1}{e^{(\epsilon-\mu)/T} - 1} \quad \Rightarrow \quad \mu \leq 0 \]

**BCS-BEC crossover**

\[ \eta < 0 \rightarrow \eta > 0, \]

small $\Delta \rightarrow$ large $\Delta,$

\[ \mu > 0 \rightarrow \mu < 0 \]
Relativistic Mean Field with Broken Universality

Lianyi He, PZ, 2007

**NJL-type model at moderate density**

\[ L = \bar{\psi} \left( i \gamma^\mu \partial_\mu - m \right) \psi + \frac{g}{4} \left( \bar{\psi} i \gamma_5 C \bar{\psi}^T \right) \left( \psi^T i C \gamma_5 \psi \right) \]

*order parameter*

\[ \Delta = \frac{g}{2} \left\langle \psi^T i C \gamma_5 \psi \right\rangle \]

*mean field thermodynamic potential*

\[ \Omega = \frac{\Delta^2}{g} - \int \frac{d^3 \vec{k}}{(2\pi)^3} \left[ E_k^+ + E_k^- - \xi_k^+ - \xi_k^- \right] \]

*fermion and anti-fermion contributions*

\[ E_k^\pm = \sqrt{\left( \xi_k^\pm \right)^2 + \Delta^2} \]

\[ \xi_k^\pm = \sqrt{k^2 + m^2} \pm \mu \]

*gap equation and number equation:*

\[ \left\{ \begin{array}{l}
- \frac{\pi}{2} \eta = \int_0^z dxx^2 \left[ \left( \frac{1}{E_x^-} - \frac{1}{\xi_x - 2\zeta^{-2}} \right) + \left( \frac{1}{E_x^+} - \frac{1}{\xi_x + 2\zeta^{-2}} \right) \right] \\
\frac{2}{3} = \int_0^z dxx^2 \left[ \left( 1 - \frac{\xi_x^-}{E_x^-} \right) - \left( 1 - \frac{\xi_x^+}{E_x^+} \right) \right]
\end{array} \right. \]

\[ \eta = \frac{1}{k_F a_s}, \quad \zeta = \frac{k_F}{m} \]

*broken universality*

*extra density dependence*
**Relativistic Mean Field BCS-BEC**

\( \mu - m \) plays the role of non-relativistic chemical potential

- \( \mu - m = 0 \): BCS-NBEC crossover
- \( \mu = 0 \): fermion and anti-fermion degenerate, NBEC-RBEC crossover

In non-relativistic case, there is only one variable \( \eta = \frac{1}{k_F} \alpha_s \), changing the density cannot induce a BCS-BEC crossover.

However, in relativistic case, the extra density dependence \( \xi = \frac{k_F}{m} \) may induce a BCS-BEC.
Fluctuations: \( G_0 G \) Scheme

Lianyi He, PZ, 2007

**bare fermion propagator**

\[
G_0^{-1}(k, \mu) = (k_0 + \mu)\gamma_0 - \vec{\gamma} \cdot \vec{k} - m
\]

**mean field fermion propagator**

\[
G^{-1}(k, \mu) = G_0^{-1}(k, \mu) - \Sigma_{mf}(k)
\]

**pair propagator**

\[
= \frac{ig}{1 - g} \quad \text{or} \quad = \frac{G_0}{g/2} + \ldots
\]

**pair feedback to the fermion self-energy**

\[
\Sigma(k) = \Sigma_{mf}(k) + \Sigma_{fl}(k) = \ldots + \ldots
\]

fermions and pairs are coupled to each other

**approximation**

\[
\Sigma_{fl}(k) \simeq -\Delta_{pg}^2 G_0(-k, \mu)
\]

the pseudogap is related to the uncondensed pairs, in \( G_0 G \) scheme the pseudogap does not change the symmetry structure
Fluctuations: BCS-NBEC-RBEC

$T_c$: critical temperature
$T^*$: pair dissociation temperature

$T < T_c$: $\Delta \neq 0$, $\Delta_{pg} \neq 0$, condensed phase

$T_c < T < T^*$: $\Delta = 0$, $\Delta_{pg} \neq 0$, normal phase with both fermions and pairs

$T > T^*$: $\Delta = 0$, $\Delta_{pg} = 0$, normal phase with only fermions

**BCS:** $\eta < 0$, $\mu > m$ **no pairs**

**NBEC:** $0 < \eta < m / k_F$, $0 < \mu < m$ **heavy pairs, no anti-pairs**

**RBEC:** $\eta > m / k_F$, $\mu \sim 0$ **light pairs, almost the same number of pairs and anti-pairs**
asymmetric nuclear matter with both np and nn and pp pairings
density-dependent contact interaction (Garrido et al, 1999)

\[ V(x, x') = u \left( 1 - \eta \left\{ \frac{\rho[(x + x')/2]}{\rho_0} \right\}^{\nu} \right) \delta(x - x'), \]

and density-dependent nucleon mass (Berger, Girod, Gogny, 1991)

by calculating the three coupled gap equations, there exists only np pairing BEC state at low density and no nn and pp pairing BEC states.
order parameters of spontaneous chiral and color symmetry breaking

\[ \sigma = \langle \bar{\psi} \psi \rangle \quad \Delta = \Delta^3 = \langle \bar{\psi}_i^C \varepsilon^{ij} \varepsilon^{a\beta j} i \gamma_5 \psi_j^\alpha \rangle \]  

color breaking from SU(3) to SU(2)

quarks at mean field and mesons and diquarks at RPA

quark propagator in 12D Nambu-Gorkov space

\[
\Psi = \left( \begin{array}{cccccc}
\bar{\psi}_u^1 & \bar{\psi}_d^C & \bar{\psi}_d^2 & \bar{\psi}_u^1 & \bar{\psi}_d^C & \bar{\psi}_d^2 \\
\bar{\psi}_u^C & \bar{\psi}_d^1 & \bar{\psi}_u^2 & \bar{\psi}_d^3 & \bar{\psi}_u^C & \bar{\psi}_d^3 \\
S_A & S_B & S_C & S_D & S_E & S_F
\end{array} \right)
\]

\[ S_I = \begin{pmatrix} G^+_I & \Xi^-_I \\ \Xi^+_I & G^-_I \end{pmatrix} \quad I = A, B, C, D, E, F \]

\[ E_k = \sqrt{k^2 + M_q^2} \quad M_q = m_0 - 2G_S \sigma \]

\[ E_\Delta^\pm = \sqrt{(E_k \pm \mu)^2 + (2G_D \Delta)^2} \]

diquark & meson polarizations

diquark & meson propagators at RPA

\[ \begin{array}{c}
\begin{array}{cccccc}
\bar{\psi}_u^1 & \bar{\psi}_d^C & \bar{\psi}_u^2 & \bar{\psi}_u^1 & \bar{\psi}_d^C & \bar{\psi}_d^2 \\
\bar{\psi}_u^C & \bar{\psi}_d^1 & \bar{\psi}_u^2 & \bar{\psi}_d^3 & \bar{\psi}_u^C & \bar{\psi}_d^3 \\
S_A & S_B & S_C & S_D & S_E & S_F
\end{array}
\end{array}
\]

\[ \prod_D \Gamma_D \quad \prod_M \Gamma_M \]
gap equations for chiral and diquark condensates at $T=0$

\[
\begin{align*}
  m - m_0 &= 8G_s m \int \frac{d^3k}{(2\pi)^3} \frac{1}{E_p} \left[ \frac{E_k - \mu_B / 3}{E^-_\Delta} + \frac{E_k + \mu_B / 3}{E^+_\Delta} + \Theta(E_k - \mu_B / 3) \right] \\
  \Delta &= 8G_d \Delta \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{E^-_\Delta} + \frac{1}{E^+_\Delta} \right]
\end{align*}
\]

to guarantee color neutrality, we introduce color chemical potential:

\[
\mu_r = \mu_g = \mu_B / 3 + \mu_8 / 3, \quad \mu_b = \mu_B / 3 - 2\mu_8 / 3
\]

there exists a BCS-BEC crossover

color neutrality speeds up the chiral restoration and reduces the BEC region

\[
\eta = \frac{G_d}{G_s}
\]
BCS-BEC in Pion Superfluidity

**meson mass, Goldstone mode**

\[ \rho(\omega, \vec{k}) = -2 \text{Im} D(\omega, \vec{k}) \]

**meson spectra function**

- **BCS-BEC crossover**
- **BCS**
- **BEC**
- **critical temperature**
- **pion superfluid**

**Gaofeng Sun, Lianyi He, PZ, 2007**
Conclusions

* **BCS-BEC crossover is a general phenomena from cold atom gas to quark matter.**

* **BCS-BEC crossover is closely related to the QCD key problems: vacuum, color symmetry, chiral symmetry, isospin symmetry ……**

* **BCS-BEC crossover in color superconductivity and pion superfluidity is not induced by simply increasing the coupling constant of the attractive interaction, but by changing the corresponding charge number.**

* there are potential applications in heavy ion collisions (at CSR/Lanzhou, FAIR/GSI, NICA/JINR and RHIC/BNL) and compact stars.

**thanks for your patience**
backups
vector meson coupling and magnetic instability

vector-meson coupling

\[ L_V = -G_V \left[ (\overline{\psi} \gamma_\mu \psi)^2 + (\overline{\psi} \gamma_\mu \gamma_5 \tau \psi)^2 \right] \]

vector condensate

\[ \rho_V = 2G_V \langle \overline{\psi} \gamma_0 \psi \rangle \]

gap equation

\[ \rho_V = 8G_V \int \frac{d^3k}{(2\pi)^3} \left[ \frac{E_k + \mu_B / 3}{E_k^+} - \frac{E_k - \mu_B / 3}{E_k^-} + \Theta(-E_k + \mu_B / 3) \right] \]

\( \eta = 1 \)

vector meson coupling slows down the chiral symmetry restoration and enlarges the BEC region.

Meissner masses of some gluons are negative for the BCS Gapless CSC, but the magnetic instability is cured in BEC region.
Beyond mean field

\[ \Delta_0 = \Delta(T = 0) \text{ is determined by the coupling and chemical potential} \]

\[ \Delta_0 = 100 - 200 \text{ MeV} \leftrightarrow \mu_q = 300 - 500 \text{ MeV} \]

- Going beyond mean field reduces the critical temperature of color superconductivity
- Pairing effect is important around the critical temperature and dominates the symmetry restored phase
**NJL with isospin symmetry breaking**

\[ L_{NJL} = \bar{\psi} \left( i \gamma^\mu \partial_\mu - m_0 + \mu_0 \gamma_0 \right) \psi + G \left( (\bar{\psi} \psi)^2 + (\bar{\psi} i \tau_1 \gamma_5 \psi)^2 \right) \]

**quark chemical potentials**

\[
\mu = \begin{pmatrix}
\mu_u & 0 \\
0 & \mu_d
\end{pmatrix} = \begin{pmatrix}
\mu_B / 3 + \mu_I / 2 & 0 \\
0 & \mu_B / 3 - \mu_I / 2
\end{pmatrix}
\]

**chiral and pion condensates with finite pair momentum**

\[
\sigma = \langle \bar{\psi} \psi \rangle = \sigma_u + \sigma_d, \quad \sigma_u = \langle \bar{u} u \rangle, \quad \sigma_d = \langle \bar{d} d \rangle
\]

\[
\pi_+ = \sqrt{2} \langle \bar{u} i \gamma_5 d \rangle = \frac{\pi e^{2i\eta \cdot \vec{x}}}{\sqrt{2}}, \quad \pi_- = \sqrt{2} \langle \bar{d} i \gamma_5 u \rangle = \frac{\pi e^{-2i\eta \cdot \vec{x}}}{\sqrt{2}}
\]

**quark propagator in MF**

\[
S^{-1}(p, \bar{q}) = \begin{pmatrix}
\gamma^\mu p_\mu - \bar{\gamma} \cdot \bar{q} + \mu_u \gamma_0 - m \\
2iG\pi \gamma_5
\end{pmatrix}
\begin{pmatrix}
2iG\pi \gamma_5 \\
\gamma^\mu k_\mu + \bar{\gamma} \cdot \bar{q} + \mu_d \gamma_0 - m
\end{pmatrix}
\]

\[ m = m_0 - 2G\sigma \]

**thermodynamic potential and gap equations:**

\[
\Omega = G(\sigma^2 + \pi^2) - \frac{T}{V} \text{Tr} \ln S^{-1}
\]

\[
\frac{\partial \Omega}{\partial \sigma_u} = 0, \quad \frac{\partial^2 \Omega}{\partial \sigma_u^2} \geq 0, \quad \frac{\partial \Omega}{\partial \sigma_d} = 0, \quad \frac{\partial^2 \Omega}{\partial \sigma_d^2} \geq 0, \quad \frac{\partial \Omega}{\partial \pi} = 0, \quad \frac{\partial^2 \Omega}{\partial \pi^2} \geq 0, \quad \frac{\partial \Omega}{\partial q} = 0, \quad \frac{\partial^2 \Omega}{\partial q^2} \geq 0
\]
mesons in RPA

meson propagator \( D \) at RPA

\[
\begin{align*}
\Pi_{mn}(k) &= i \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \left( \Gamma_m^* S(p+k) \Gamma_n S(p) \right) \\
\Gamma_m &= \begin{cases} 
1, & m = \sigma \\
i \tau_+ \gamma_5, & m = \pi_+ \\
i \tau_- \gamma_5, & m = \pi_- \\
i \tau_5 \gamma_5, & m = \pi_0
\end{cases}
\end{align*}
\]

pole of the propagator determines meson masses \( M_m \)

\[
\begin{pmatrix}
1 - 2G\Pi_{\sigma\sigma}(k) & -2G\Pi_{\sigma\pi^+}(k) & -2G\Pi_{\sigma\pi^-}(k) & -2G\Pi_{\sigma\pi_0}(k) \\
-2G\Pi_{\pi^+\sigma}(k) & 1 - 2G\Pi_{\pi^+\pi^+}(k) & -2G\Pi_{\pi^+\pi^-}(k) & -2G\Pi_{\pi^+\pi_0}(k) \\
-2G\Pi_{\pi^-\sigma}(k) & -2G\Pi_{\pi^-\pi^+}(k) & 1 - 2G\Pi_{\pi^-\pi^-}(k) & -2G\Pi_{\pi^-\pi_0}(k) \\
-2G\Pi_{\pi_0\sigma}(k) & -2G\Pi_{\pi_0\pi^+}(k) & -2G\Pi_{\pi_0\pi^-}(k) & 1 - 2G\Pi_{\pi_0\pi_0}(k)
\end{pmatrix}
\begin{pmatrix} k_0 \equiv M_m \end{pmatrix} = 0
\]

mixing among normal \( \sigma, \pi^+, \pi^- \) in pion superfluid phase,
the new eigen modes \( \overline{\sigma}, \overline{\pi}^+, \overline{\pi}^- \) are linear combinations of \( \sigma, \pi^+, \pi^- \)
In NJL, Linear Sigma Model and Chiral Perturbation Theory, there is no remarkable difference around the critical point.

Analytic result:
Critical isospin chemical potential for pion superfluidity is exactly the pion mass in the vacuum:

$$\mu_i^c = m_\pi$$

Pion superfluidity phase diagram in \(\mu_i - \mu_B\) plane at \(T=0\):

- \(\mu_i\): average Fermi surface
- \(\mu_B(n_B)\): Fermi surface mismatch

Homogeneous (Sarma, \(\vec{q} = 0\)) and inhomogeneous pion superfluid (LOFF, \(\vec{q} \neq 0\))

Magnetic instability of Sarma state at high average Fermi surface leads to the LOFF state.