Transport properties of light meson gases and chiral symmetry restoration

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Quick review of the diagramatic method for calculating transport coefficients in quantum field theory

A diagramatic calculation of the shear and bulk viscosities for a meson gas in ChPT.

The role of resonances and chiral symmetry restoration in TC, KSS bound, trace anomaly, sum rules, comparison with other results for the hadron gas, ...

Conclusions
Diagramatic method for calculating transport coefficients

In presence of viscosities, the energy-momentum tensor of the fluid is modified. To first order in gradients,

\[ T_{ij} = P \delta_{ij} + \eta \left( \partial_i U_j + \partial_j U_i + \frac{2}{3} \delta_{ij} \nabla \cdot U \right) - \zeta \delta_{ij} \nabla \cdot U, \]

where \( \eta \) is the shear viscosity, \( \zeta \) is the bulk viscosity, \( P \) is the pressure, \( \rho \) is the energy density, \( P \) is the pressure, \( \nabla \cdot U \) is the fluid velocity, \( \nabla \cdot U \) is the thermodynamic force, \( U \) is the fluid velocity, \( \nabla \cdot U \) is the thermodynamic force, \( \delta_{ij} \) is the Kronecker delta, and \( \nabla \cdot U \) is the thermodynamic force.

In Linear Response Theory (LRT):

\[ \eta = \frac{1}{20} \lim_{q^0 \to 0+} \lim_{|q| \to 0+} \frac{\partial \rho_\eta(q^0, q)}{\partial q^0}, \quad \zeta = \frac{1}{2} \lim_{q^0 \to 0+} \lim_{|q| \to 0+} \frac{\partial \rho_\zeta(q^0, q)}{\partial q^0}, \]

with

\[ \rho_\eta(q^0, q) = 2 \text{Im} \int d^4x \ e^{i q \cdot x} \theta(t) \langle [\hat{\pi}_{ij}(x), \hat{\pi}_{ij}^\dagger(0)] \rangle, \quad \rho_\zeta(q^0, q) = 2 \text{Im} \int d^4x \ e^{i q \cdot x} \theta(t) \langle [\hat{P}(x), \hat{P}(0)] \rangle. \]

where \( \hat{\pi}_{ij} \equiv T_{ij} - g_{ij} T^l_l / 3, \quad \hat{P} \equiv -T^l_l / 3 - v_s^2 T_{00} - \mu N^0 \cdot \)

\( g_{\mu\nu} = \text{diag}(+, -, -, -) \) and \( v_s \) is the speed of sound in the fluid.
Consider for instance $\lambda \phi^4$: to one-loop order,

$$\hat{O} \sim G_A(q^0, q)G_R(q^0, q) \simeq \frac{\pi}{4E_q^2\Gamma_q} \left[ \delta(q^0 - E_q) + \delta(q^0 + E_q) \right]$$

if $\Gamma_q$ is small

★ Particle width: $\Gamma \sim \text{Im} \sim \lambda^2$

Therefore, in $\lambda \phi^4$ a resummation is necessary:

(ladder diagram) $= \mathcal{O}(1/\lambda^2)$

(bubble diagram) $= \mathcal{O}(1/\lambda^{n+1})$
Chiral Perturbation Theory (ChPT)

- We’re interested in the (non-perturbative) low-energy regime of QCD, i.e. $E \lesssim 1 \text{ GeV}$ and $T \lesssim 200 \text{ MeV}$. There, chiral symmetry is spontaneously broken:

  $$\chi \equiv SU(3)_L \times SU(3)_R \equiv SU(3)_V \times SU(3)_A \rightarrow SU(3)_V.$$

- In that regime, the degrees of freedom are the corresponding Goldstone bosons: pions, kaons and etas.

- Chiral symmetry acts non-linearly on the Goldstone bosons:

  \[
  U(x) \equiv \exp \left( \frac{i \phi(x)}{F_0} \right), \quad \phi(x) = \sum_{a=1}^{8} \lambda_a \phi_a(x) \]

  \[
  \Rightarrow \quad [Q^V_a, \phi_b] = i f_{abc} \phi_c, \quad [Q^A_a, \phi_b] = g_{ab}(\phi),
  \]

- ChPT lagrangian: The most general expansion in terms of derivatives of the field $U(x)$ and masses that fulfills all the symmetries of QCD:

  $$\mathcal{L}_{\text{ChPT}} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots \quad (\text{infinite \# of terms})$$
Leading order:

\[ \mathcal{L}_2 = \frac{F_0^2}{4} \text{Tr}\{(\nabla_\mu U)(\nabla^\mu U)^\dagger\} + \frac{F_0^2}{4} \text{Tr}\{\chi U^\dagger + U \chi^\dagger\}. \]

Next-to-leading order:

\[ \mathcal{L}_4 = L_1 \left(\text{Tr}\{(\nabla_\mu U)(\nabla^\mu U)^\dagger\}\right)^2 + L_2 \text{Tr}\{(\nabla_\mu U)(\nabla_\nu U)^\dagger\} \text{Tr}\{(\nabla^\mu U)(\nabla^\nu U)^\dagger\} \\
+ L_3 \text{Tr}\{(\nabla_\mu U)(\nabla^\mu U)^\dagger(\nabla_\nu U)(\nabla^\nu U)^\dagger\} + L_4 \text{Tr}\{(\nabla_\mu U)(\nabla^\mu U)^\dagger\} \text{Tr}\{\chi U^\dagger + U \chi^\dagger\} \\
+ L_5 \text{Tr}\{(\nabla_\mu U)(\nabla^\mu U)^\dagger(\chi U^\dagger + U \chi^\dagger)\} + L_6 \left(\text{Tr}\{\chi U^\dagger + U \chi^\dagger\}\right)^2 \\
+ L_7 \left(\text{Tr}\{\chi U^\dagger - U \chi^\dagger\}\right)^2 + L_8 \text{Tr}\{U \chi^\dagger U \chi^\dagger + \chi U^\dagger \chi U^\dagger\} \\
- iL_9 \text{Tr}\{f_{\mu\nu}^R(\nabla_\mu U)(\nabla_\nu U)^\dagger + f_{\mu\nu}^L(\nabla_\mu U)^\dagger(\nabla^\nu U)\} + L_{10} \text{Tr}\{U f_{\mu\nu}^L U^\dagger f_{\mu\nu}^R\} \\
+ H_1 \text{Tr}\{f_{\mu\nu}^R f_{\mu\nu}^R + f_{\mu\nu}^L f_{\mu\nu}^L\} + H_2 \text{Tr}\{\chi \chi^\dagger\}. \]

The constants \(F_0, B_0, L_1, L_2, L_3, L_4, L_5, L_6, L_7, L_8, L_9, L_{10}, H_1, H_2\) are energy- and temperature-independent, and are determined experimentally.
Dimension, $D$, of a Feynman diagram:

Re-scaling: \[
\begin{align*}
    p_i &\mapsto t p_i \\
    m_q &\mapsto t^2 m_q
\end{align*}
\] \[\Rightarrow \quad \mathcal{M}(t p_i, t^2 m_q) = t^D \mathcal{M}(p_i, m_q) .
\]

amplitude of the diagram

Weinberg’s theorem:

\[D = 2 + \sum_n N_n(n - 2) + 2L .\]

number of vertices from $\mathcal{L}_n$

number of loops

Perturbation theory with respect to the scales: $\Lambda_X \sim 1$ GeV (for momenta), $\Lambda_T \sim 200$ MeV (for temperatures).
**Ladder diagrams:**

Lines with $\Gamma = 0$. These loops (rungs) give a perturbative contribution $\sim X$.

If we only consider constant vertices:

- $T \ll M_\pi$, $Y \sim \sqrt{\frac{M_\pi}{T}}$, $X \sim \frac{1}{Y} \left(\frac{M_\pi}{4\pi F_\pi}\right)^2$.
- $T \approx M_\pi$, $Y \sim 1$, $X \lesssim \left(\frac{M_\pi}{4\pi F_\pi}\right)^2$.

Each pair of lines with $\Gamma \neq 0$ and equal momentum give a pinching pole contribution $\sim Y$.

If $T \gtrsim M_\pi$, $X \sim 1$, derivative vertices start to dominate ⇒ a large number of diagrams become important.

**Bubble diagrams:**

= $\mathcal{O}(B^{(0)} X^{n-1})$

Contribution from a simple bubble $\sim Y$.

Weinberg's theorem does not provide the (non-perturbative) right order for TC at low $T$: $\mathcal{O}(p^{2n}) \gg \mathcal{O}(p^{4n})$.

In principle, for low $T$, the leading order is a one-loop diagram.
Thermal width in ChPT

- Pion thermal width:
  \[ \Gamma \sim \text{Im} \ T_{\pi\pi} \]

Dilute gas approximation: \[ \Gamma(k_1) = \frac{1}{2} \int \frac{d^3k_2}{(2\pi)^3} \ e^{-\beta E_2} \sigma_{\pi\pi} \nu_{\text{rel}} (1 - \nu_1 \cdot \nu_2) \]

Scattering cross section: \[ \sigma_{\pi\pi}(s) \simeq \frac{32\pi}{3s} \left[ |t_{00}(s)|^2 + 9|t_{11}(s)|^2 + 5|t_{20}(s)|^2 \right] . \]

- ChPT violates the unitarity condition for high \( p \): \( S^\dagger S = 1 \Rightarrow \text{Im} \ t_{IJ}(s) = \sigma(s)|t_{IJ}(s)|^2 \)
  with \( \sigma(s) \equiv \sqrt{1 - 4M_{\pi}^2/s} \).

Because partial waves are essentially polynomials in \( p \): \( t_{IJ}(s) = t_{IJ}^{(1)}(s) + t_{IJ}^{(2)}(s) + O(s^3) \).

- The Inverse Amplitude Method (IAM):
  \[ t_{IJ}(s) \simeq \frac{t_{IJ}^{(1)}(s)}{1 - t_{IJ}^{(2)}(s; T)/t_{IJ}^{(1)}(s)} . \]

It verifies the unitarity condition exactly and reproduces resonant states.
Behavior of the $\sigma$ and $\rho$ resonances in medium

- Finite temperature:

\[ \Gamma (\text{MeV}) \]

\[ T = 0 \]

\[ T = 200 \text{ MeV} \]

\[ T = 250 \text{ MeV} \]

\[ M (\text{MeV}) \]

\[ \sigma \]

\[ \rho \]

- Finite nuclear density:

We can encode approximately nuclear density effects into $F_{\pi}$:

\[
\frac{F_{\pi}^2(\rho)}{F_{\pi}^2(0)} \approx \left( 1 - \frac{\sigma_{\pi N}}{M_{\pi}^2 F_{\pi}^2(\rho)} \right) + \mathcal{O}(M_{\pi})
\]

\[
\approx \left( 1 - 0.35 \frac{\rho}{\rho_0} \right) + \mathcal{O}(M_{\pi})
\]

where $\sigma_{\pi N} \simeq 45 \text{ MeV}$, and $\rho_0 \simeq 0.17 \text{ fm}^{-3}$.


For an improved study of temperature and nuclear density effects for the $\sigma$ using the Bethe-Salpeter eq. see

Cabrera, Gomez Nicola, & DFF, EPJC 61 879 (2009).

entropi density

\[ \eta = v \omega + \frac{1}{2} \Gamma \]

Then for \( T \ll M_\pi \), \( \eta, \zeta \sim \sqrt{T} \).

Shear viscosity of a pion gas

Results:

Sound attenuation length:

\[ \omega = v_s k + \frac{1}{2} i \Gamma_s k^2 \]

without \( \zeta \),

\[ \Gamma_s \approx \frac{4\eta}{3sT} \], \quad \Gamma_s(T = 180 \text{ MeV}) \approx 1.1 \text{ fm}. \]

A value \( \eta/s < 0.24 \) is necessary to explain the current data on elliptic flow.

By KT: \( \eta \sim M_\pi v n l \), but \( l \sim \frac{1}{\sigma_{\pi\pi} n} \).

AdS/CFT bound:
A minimum near $T_c$ for a pion gas:

By KT: $\eta \sim mnvl \sim \epsilon \tau$, and $s \sim n$.

\[ \Rightarrow \frac{\eta}{s} \sim E \tau \gtrsim 1 \text{ (uncertainty principle)} \]

$\tau \sim \frac{1}{\Gamma} \Rightarrow \frac{\eta}{s}$ increases at high $T$

Large $N_c$: $\zeta/s = \begin{cases} \mathcal{O}(N_c^2), & T \ll M_\pi \\ \mathcal{O}(1), & T \to \infty \end{cases}$

Full hadron resonance gas:

Noronha-Hostler, Noronha, & Greiner, arXiv: 0811.1571

\[ \eta \sim \alpha \sum_i n_i \langle p \rangle_i \lambda_i \]

$\mathcal{O}(1)$
QCD trace anomaly: 
\[ \partial_\nu J^\nu_{\text{dil}} = T^\mu_\mu = \frac{\beta(g)}{2g} G^a_{\mu\nu} G^{a\mu\nu}_\alpha + (1 + \gamma(g)) \bar{q}Mq \]

related to bulk viscosity: 
\[ \zeta = \frac{1}{9} \lim_{\omega \to 0^+} \frac{1}{\omega} \int_0^\infty dt \int d^3x \ e^{i\omega t} \langle [\hat{T}^\mu_\mu(x), \hat{T}^\nu_\nu(0)] \rangle = \frac{\pi}{9} \lim_{\omega \to 0^+} \frac{\rho_{\theta\theta}(\omega, 0)}{\omega} \]

Sum rule: 
\[ \int_{-\infty}^\infty d\omega \frac{\rho_{\theta\theta}(\omega, 0)}{\omega} = -\left(4 - T \frac{\partial}{\partial T}\right) \langle \theta \rangle_T = T^5 \frac{\partial}{\partial T} \frac{(\epsilon - 3P)^*}{T^4} + 16|\epsilon_v| \]

\[ \langle \theta \rangle_T \equiv \langle T^\mu_\mu \rangle_T = \epsilon - 3P \]

Ansatz: 
\[ \frac{\rho_{\theta\theta}(\omega, 0)}{\omega} = \frac{9\zeta}{\pi} \frac{\omega^2_0}{\omega^2_0 + \omega^2}, \quad \omega_0 \sim 1 \text{ GeV} \]

\[ \Rightarrow \quad \zeta(T) = \frac{1}{9\omega_0(T)} \left[ T^5 \frac{\partial}{\partial T} \frac{\langle \theta \rangle_T - \langle \theta \rangle_0}{T^4} + 16|\epsilon_v| \right]. \]

Kharzeev & Tuchin, JHEP 0809:093, 2008
Kharsch, Kharzeev & Tuchin, PLB 663, 217 (2008)

An increase of \( \zeta \) near \( T_c \)?
There is a recent modification of the sum rule (corresponding to exchanging the external frequency and momentum limits): \[ 3(\epsilon + P)(1 - 3c_s^2) - 4(\epsilon - 3P) = \frac{2}{\pi} \int \frac{d\omega}{\omega} [\rho_\zeta(\omega) - \rho_\zeta^{T=0}(\omega)] . \]

An ansatz for \( \rho_\zeta(\omega) - \rho_\zeta^{T=0}(\omega) \) near \( \omega = 0 \) might miss important information from the high-\( \omega \) region:

\[ \frac{\rho_\zeta^*(\omega)}{\omega} \sim \frac{1}{\omega \log^2 \omega} \]

Even in \( \lambda \phi^4 \), the correlation between \( \zeta \) and \( T_\mu^\mu \) is not direct:

\[ T \ll m : \begin{cases} T_\mu^\mu \sim T^{3/2} m^{5/2} e^{-m/T} \\ \zeta \sim e^{2m/T} m^6 / \lambda^4 T^3 \end{cases} , \quad m \equiv 0 : \begin{cases} T_\mu^\mu \sim \beta(\lambda)T^4 \\ \zeta \sim \lambda T^3 \log^2 \lambda \end{cases} \]

\[ \text{Romatschke & Son, arXiv:0903.3946} \]

\[ \text{Caron-Huot, PRD 79, 125009 (2009)} \]

\[ \text{Jeon & Yaffe, PRD 53, 5799 (1996)} \]
From the ChPT pressure:  
\[ \langle T^\mu_\mu \rangle_T = T^5 \frac{d}{dT} \left( \frac{P}{T^4} \right) \]

For \( m_u = m_d = 0 \):

\[ \langle T^\mu_\mu \rangle^* = \frac{\pi^2}{270} T^8 \frac{1}{F^4} \left( \ln \frac{A_p}{T} - \frac{1}{4} \right), \quad A_p \sim 400 \text{ MeV} . \]

Hadron Resonance Gas vs Lattice (2+1 q’s):

HRG approximation: all the resonances in the PDB up to 2 GeV are included, 1026 in total, introduced as free states.

\[ \Delta \equiv \frac{\epsilon - 3P}{T^4} = \sum_{i=1}^{1026} \frac{\epsilon_i - 3P_i}{T^4} \]

\[ = \sum_{i=1}^{1026} \frac{g_i}{2\pi^2} \sum_{k=1}^{\infty} (-\eta)^{k+1} \frac{(\beta m_i)^3}{k} K_1(k\beta m_i) \]
The role of in-medium resonances

- **Trace anomaly in the Virial Gas Approximation (dilute gas):**

\[
\beta P = \sum_i \left( B_i^{(1)} \xi_i + B_i^{(2)} \xi_i^2 + \sum_{j \geq i} B_{\text{int}} \xi_i \xi_j + \ldots \right)
\]

\[
\xi_i \equiv e^{\beta(\mu_i - m_i)} , \quad B_i^{(n)} = \frac{g_i \eta_i^{n+1}}{2 \pi^2 n} \int_0^\infty dp \ p^2 e^{-n\beta(E_i - m_i)}
\]

\[
B_{ij}^{\text{int}} = \frac{e^{\beta(m_i + m_j)}}{2\pi^3} \int_{m_i + m_j}^\infty dE \ E^2 K_1(\beta E) \sum_{I,J,S} (2I + 1)(2J + 1) \delta_{IJS}^{ij}(E)
\]

Therefore, we would not expect a large effect on \( \zeta \) due to in-medium modification of the \( \sigma \) and \( \rho \) resonances.

- **Bulk viscosity in the Linear Sigma Model:**

\[
\zeta \propto \frac{\Gamma_\sigma}{m_\sigma^2}
\]

*Paech & Pratt, PRC 74, 014901 (2006)*

![Graph showing the role of in-medium resonances](image-url)
**Bulk viscosity of a gas of pions**

- **Heat capacity and speed of sound (ChPT):**

  - \( O(T^8), \ m_q = 0 \)
  - \( O(T^6) \)
  - \( O(T^4) \)

- **Lattice (2+1 flavors):**

  - Gomez Nicola & DFF, PRL 102, 121601(2009)

- **Bulk viscosity (including only 2 → 2 processes):**

  - \( \zeta^{(0)} \)
  - \( \frac{1}{(140 \text{ MeV})^3} \)
  - unit. \( O(p^4) \)
  - \( F_\pi = 50 \text{ MeV} \)

\[
\zeta^{(0)} = \int_0^\infty dp \frac{3p^2(p^2/3 - c_s^2E_p^2)^2}{4\pi^2TE_p^2\Gamma_p} n_B(E_p)[1 + n_B(E_p)]
\]

- \( T \ll M_\pi : \ \zeta^{(0)} \simeq 0.36 \eta^{(0)} \)
- \( T \simeq M_\pi : \ \zeta^{(0)} \sim 10^{-1} \eta^{(0)} \)
- \( T \gg M_\pi : \ \zeta^{(0)} \sim \left( \frac{1}{3} - v_s^2 \right)^2 \eta^{(0)} \)
**Bulk viscosity of a gas of pions**

- **The $\zeta/s$ quotient near $T_c$ and the speed of sound:** By KT: $\zeta \sim mnvl \left(\frac{1}{3} - v_s^2\right)^2$

  According to this, for the full hadron resonance gas near $T_c$:

  $$\frac{\zeta}{s} (T = T_c) \approx A \left(\frac{1}{3} - c_s^2\right)^2 \approx 0.3 \gtrsim \frac{\eta}{s} (T_c)$$

  approximately independent of the number of degrees of freedom

- **$\zeta/s$ for the massless pion gas in KT:**

  $\zeta/s$ for the massless pion gas in KT:

  \[ T_{\mu\nu} = g_\pi \int \frac{d^3p}{(2\pi)^3} \frac{f_{eq}}{E_p} \left[ p_\mu p_\nu (1 + g_1) + \frac{g_2 g_{\mu\nu}}{\beta^2} + \frac{g_3 U_\mu U_\nu}{\beta^2} \right] \]

  \[ \frac{\zeta}{s} (T = T_c) \gtrsim 3 ? \]

  Even a bigger effect might come from vertex corrections.

*Chen & Wang, PRC 79, 044913 (2009)*
Conclusions

- The ChPT diagramatic method presented allows to easily obtain the functional form of transport coefficients at low $T$, including the in-medium evolution of resonances.

- The method can be extended to include other degrees of freedom: kaons, etas, baryons, and the corresponding resonances.

- Resonances make the quotient $\eta/s$ for a pion gas fulfill the KSS bound and reach a minimum near $T_c$.

- There are several indications that there is a maximum of the bulk viscosity near $T_c$ driven by the maximum of the trace anomaly.

- Some estimations suggest that $\zeta/s$ might be larger than $\eta/s$ near $T_c$.

- Several effects contribute to a large bulk viscosity: small speed of sound, vertex corrections, and resonances.
Backup slides
Consider a small deviation from equilibrium:  
\[ f(x, p) = f_{eq}(x, p) + \delta f_{out}(x, p) \]

\[ \delta f_{out}(x, p) \equiv f_{eq}(x, p)[1 + f_{eq}(x, p)]\phi(x, p) \]

By linearizing the transport equation with respect to \( \phi \):

\[ p^\mu \partial_\mu f_{eq}|_{lin} = \beta p^0[q_\xi(|p|) \nabla \cdot U + q_\eta(|p|) \hat{p}_i \hat{p}_j \partial_i \hat{U}_j] f_{eq}(1 + f_{eq}) , \]

\[ f_{eq}(x, p) = \frac{1}{e^{\beta p_\mu U^\mu} - 1} \]

\[ C[f_1]|_{lin} = \frac{1}{2(2\pi)^3} f_{1,eq} \int \frac{d^3p_2}{p_2^0} \frac{d^3p_1'}{p_1'^0} \frac{d^3p_2'}{p_2'^0} f_{2,eq}(1 + f_{1,eq}(1 + f_{2,eq})[\phi_1 + \phi_2 - \phi_1 - \phi_2] \]

\[ \times \delta^{(4)}(p_1 + p_2 - p_1' - p_2') |\langle p_2', p_1'|T|p_1, p_2\rangle|^2 \equiv f_{1,eq} C[\phi] \]

with \( \partial_i \hat{U}_j \equiv \partial_i U_j + \partial_j U_i + \frac{2}{3} \delta_{ij} \nabla \cdot U \).

Then \( \phi \) must be of the form:

\[ \phi = A(|p|) \nabla \cdot U + B(|p|) \hat{p}_i \hat{p}_j \partial_i \hat{U}_j \]

expressions for the shear and bulk viscosities
The kinetic theory approach to calculate transport coefficients

Then we can write the transport equation for each type of deviation from equilibrium symbolically as:

\[ S^a = C[\chi^a], \quad \text{and} \quad \eta, \zeta \propto \langle S^a | \chi^a \rangle, \]

where

\[ S_{\eta}^{ij} \equiv -T q_\eta(|p|) \hat{p}^i \hat{p}^j f_{eq}(1 + f_{eq}), \quad \chi_{\eta}^{ij} \equiv \hat{p}^i \hat{p}^j B(|p|) \]

\[ S_{\zeta} \equiv -T q_\zeta(|p|) f_{eq}(1 + f_{eq}), \quad \chi_{\zeta} \equiv A(|p|) \]

\[ \langle f | g \rangle \equiv \beta^3 \int \frac{d^3 p}{(2\pi)^3} f(p) g(p) \]

Finally,

\[ \eta = \frac{2}{15} \langle S_\eta | \hat{C}^{-1} | S_\eta \rangle, \quad \zeta = \langle S_\zeta | \hat{C}^{-1} | S_\zeta \rangle. \]
Shear viscosity in $\lambda \phi^4$

**Bubble diagrams** can be easily resummed:

$$\sum_{n=1}^{\infty} \mathcal{V}^{(0)} = 0,$$

because of rotational invariance ($\mathcal{V}_{ij}^{(0)} = \partial_i \phi \partial_j \phi + \frac{1}{3} \delta_{ij} \partial_k \phi \partial^k \phi$).

**The resummation of ladder diagrams** instead implies to solve an integral equation:

| $\mathcal{V}$ | $\mathcal{V}^{(0)}$ | $\mathcal{M} \mathcal{F}$ |

$$|\mathcal{V}\rangle = |\mathcal{V}^{(0)}\rangle + \mathcal{K}|\mathcal{V}\rangle, \quad \mathcal{K} \equiv \mathcal{M} \mathcal{F}.$$  

$$\eta = \frac{\beta}{10} \lim_{\omega \to 0^+} \lim_{|p| \to 0^+} \langle \mathcal{V}^{(0)}|\hat{\mathcal{F}}|\mathcal{V}\rangle [1 + \mathcal{O}(\lambda)].$$
For $\zeta$, **bubble diagrams** cannot be neglected:

$$
\sum_{n=1}^{\infty} \mathcal{V}^{(0)}_{n} = \mathcal{V}^{(0)} + \mathcal{O}(\lambda),
$$

and $\mathcal{V}^{(0)} \sim \mathcal{O}(\lambda)$.

Because the real part of a bubble does not contain pinching poles.

In this case, the resummation of **ladder diagrams** involves more complicated rungs:

$$
\zeta = \beta \lim_{\omega \to 0^+} \lim_{|p| \to 0^+} \langle \mathcal{V}^{(0)} | \hat{F} | \mathcal{V} \rangle [1 + \mathcal{O}(\lambda)] .
$$

(Contribution from number-changing processes)
Consider for instance $\lambda \phi^4$. For $T \gg m$, apparently the KT treatment is not applicable:

$$l_{\text{free}} \sim \frac{1}{T} \lesssim l_{\text{Compton}}(T = 0)$$

However, for a weakly coupled theory, at an arbitrary temperature there is an effective KT description:

$$l_{\text{free}} \sim \frac{1}{\lambda^2 T} > l_{\text{Compton}}(T) \sim \frac{1}{\sqrt{\lambda T}}$$

Essentially, one identifies $A$ and $B$ in the KT description with the effective vertices of the diagramatic analysis, and the rung with the collision operator $\hat{C}$.

In the dispersion relation of the effective quanta enters the thermal mass instead of the vacuum mass.

Scattering amplitudes are evaluated using thermal propagators.

$$T^{\mu \nu}(x) \equiv T_{\text{eq}}^{\mu \nu} - \int \frac{d^3 p}{(2\pi)^3 E_p} \left( p^\mu p^\nu - U^\mu U^\nu T^2 \frac{\partial^2 m_{\text{th}}}{\partial T^2} \right) f_{\text{eq}}(1 + f_{\text{eq}}) \phi .$$
In order to calculate a transport coefficient, we need to invert the collision operator: 
\[ \eta, \zeta \propto \langle S|\hat{C}^{-1}|S \rangle. \]

\( \hat{C} \) has one exact zero mode corresponding to energy conservation, \( |E_0\rangle \), and an approximate one, \( |N_0\rangle \), corresponding to the particle-number conserving terms in \( \hat{C} \). This is not important for \( \eta \) (because \( \langle E_0, N_0|S_\eta \rangle = 0 \)), but it is for \( \zeta \):

\[ |E_0\rangle \] is not problematic, we simply consider the vector space orthogonal to it (since \( |E_0\rangle \) is not actually a departure from equilibrium).

Since \( \hat{C} \) is hermitian, let’s consider an orthonormal basis of eigen-states:

\[ |\chi\rangle = \sum_n \chi_n |f_n\rangle, \quad \text{with} \quad |f_0\rangle \equiv |N_0\rangle \]

\[ \zeta \propto \langle S_\zeta |\chi\rangle = \langle S_\zeta |\hat{C}^{-1}|S_\zeta \rangle = \sum_n S_n^\zeta C_n^{-1} S_n^\zeta = \sum_{n \neq 0} S_n^\zeta \frac{1}{C_n} S_n^\zeta + S_n^\zeta \frac{1}{\delta C_0 (n - \text{changing})} S_n^\zeta, \]

with \( C_n = C_n(\text{cons}) + \delta C_n(\text{n - changing}) \).

It dominates in QCD at high \( T \).

It dominates in \( \lambda \phi^4 \) at any \( T \).
Bethe-Salpeter equation:

\[
 T = [1 - VG]^{-1} V .
\]

Factorization on the mass shell in vacuum:

Finite temperature:

Pion self-energy:

\( \text{O}(0) \) \( \text{O}(\rho) \) \( \text{O}(\rho^2) \)

The factorization on the mass shell is also fulfilled at finite temperature and density.
Finite temperature results:

Finite temperature and density results:
**Electrical conductivity of a pion gas**

- **Definition:** \( j^i = \sigma E_{\text{ext}}^i \)

- **Kubo's formula:**

\[
\sigma = -\frac{1}{6} \lim_{q^0 \to 0^+} \lim_{|q| \to 0^+} \frac{\partial \rho_\sigma(q^0, q)}{\partial q^0}, \quad \rho_\sigma(q^0, q) = 2 \text{Im} \int d^4 x \, e^{i q \cdot x} \theta(t) \langle [\hat{J}_i(x), \hat{J}_i(0)] \rangle.
\]

- **Results:**

According to kinetic theory: \( \sigma \sim \frac{e^2 n_{\text{ch}} \tau}{M_\pi}, \) but \( \tau \sim 1/\Gamma, \) and \( \Gamma \sim n v \sigma_{\pi \pi}. \)

For \( T \ll M_\pi, n \sim (M_\pi T)^{3/2} e^{-M_\pi / T}, \)
\( v \sim \sqrt{T / M_\pi}, \) and \( \sigma_{\pi \pi} \) is a constant, \( \Rightarrow \)
\( \sigma \sim 1/\sqrt{T}. \)

\( T \ll M_\pi: \quad \sigma^{(0)} \simeq 15 \frac{e^2 F_\pi^4}{T^{1/2} M_\pi^{5/2}}. \)
Thermal conductivity of a gas of pions

- **Definition:** \( T^{i0} - hN^i = \kappa \frac{T^2}{\hbar} \partial_i \left( \frac{\mu}{T} \right) \)

- **Kubo’s formula:**
  \[
  \kappa = -\frac{\beta}{6} \lim_{q^0 \to 0^+} \lim_{|q| \to 0^+} \frac{\partial \rho_\kappa(q^0, q)}{\partial q^0}, \quad \rho_\kappa(q^0, q) = 2 \text{Im} \int d^4x \ e^{iq \cdot x} \theta(t) \langle [\hat{T}_i(x), \hat{T}^i(0)] \rangle.
  \]

- **Results:**

From KT: \( \kappa \sim T^{-1}(\bar{e} - h)lv. \)

For \( T \ll M_\pi, \bar{e} \sim M_\pi, h \sim 5T/2 + M_\pi, \quad \Rightarrow \kappa \sim T^{1/2}. \)

\[
T \ll M_\pi : \quad \kappa^{(0)} \simeq 63 \frac{T^{1/2} F_\pi^4}{M_\pi^{5/2}}
\]