Automatic One-Loop Computation In Quarkonium Decay and Production Within NRQCD Framework

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Outline

- Introduction to automatic one-loop computation
  - ➜ Traditional automatic one-loop computation
  - ➜ The method of region expansion
- Automatic one-loop computation in quarkonium decay and production within NRQCD framework

➜	$e^+e^- \to J/\psi + \chi_{cJ}$	α_s	JHEP10(2011)141
➜	$e^+e^- \to J/\psi + \eta_c$	$\alpha_s v^2$	PRD 85, 114018 (2012)
➜	$J/\psi \to 3\gamma$	$\alpha_s v^2$	arXiv:1210.6337
➜	$W \to B_c + \gamma$	α_s	In Preparation
➜	$\Upsilon \to J/\psi + \chi_{cJ}$	α_s	In Preparation

- Summary and conclusions

Traditional Automatic One-Loop Computation

- Automatically generate Feynman diagrams and amplitudes
  FEYNARTS, QGRAF, $\cdots$

$$\mathcal{T}^{\mu_1\cdots\mu_p} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk\frac{k^{\mu_1}\cdots k^{\mu_p}}{D_0D_1D_2\cdots D_{n-1}}$$

$$D_i = (k+r_i)^2 - m_i^2 + i\varepsilon$$

- Reduce the tensor integrals to scalar integrals

$$\begin{aligned}
A_0(m_0^2) &= \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk\frac{1}{k^2-m_0^2}\\
B_0(r_{10}^2,m_0^2,m_1^2) &= \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk\prod_{i=0}^{1}\frac{1}{(k+r_i)^2-m_i^2}\\
C_0(r_{10}^2,r_{12}^2,r_{20}^2,m_0^2,m_1^2,m_2^2) &= \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk\prod_{i=0}^{2}\frac{1}{(k+r_i)^2-m_i^2}\\
D_0(r_{10}^2,r_{12}^2,r_{23}^2,r_{30}^2,r_{20}^2,r_{13}^2,m_0^2,m_1^2,m_2^2,m_3^2) &= \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk\prod_{i=0}^{3}\frac{1}{(k+r_i)^2-m_i^2}
\end{aligned}$$

Traditional Automatic One-Loop Computation

- Reduce the tensor integrals to scalar integrals

  ➜ Generally, the reduction can be achieved by the function: `PaVeReduce` or `OneLoop` in the package FeynCalc.

  ➜ Break down for the cases with zero Gram determinant.

- Calculate the scalar integrals, manually or automatically

- Substitute the computed scalar integrals into the reduced amplitudes to get the results for further processing

  ➜ To expand the relative momentum q and project S- or P- wave

  ➜ To expand the amplitudes at the asymptotic region

  ➜ $\ldots$

The Method of Region Expansion in NRQCD

- Expand relative momentum q Before or After loop integrals?
  - ➜ Generally, we expand q after performing the loop integration, and then project the S- or P- waves for the quarkonium.
  - ➜ We can also expand q before performing the loop integration, as long as only the hard region concerned, according to the method of region expansion.
- Advantage and shortcoming of region expansion
  - ➜ Advantage: The calculation can be greatly simplified, and make some impossible calculation to be feasible.
  - ➜ Shortcoming: We can not look into the other regions than the hard one, e.g., potential region or soft region.
  - ➜ If the NRQCD factorization is valid, it will be safe to use the method of region expand to compute the short-distance coefficients, which correspond to the hard region.

The Basic Tools For The Computation

- `$Apart`: Decompose the propagators to independent ones

  Comput. Phys. Commun. **183**, 2158 (2012) [arXiv:1204.2314 [hep-ph]].

$$\texttt{\$Apart}\left[\frac{1}{x(x+a)(x+y+b)},\{x,y\}\right]=\frac{1}{ax(x+y+b)}-\frac{1}{a(x+a)(x+y+b)}$$

Generally

$$\prod_{i=1}^{N} e_i^{n_i}=\sum_j f_j \prod_{i=1}^{N_j} e_{k_{ji}}^{n_{ji}}$$

Physical Example

$$\int \frac{d^n k}{(2\pi)^n}\frac{k\cdot p_1 k\cdot p_2}{D_0\ D_1\ D_2}$$

with

$$D_0=k^2,\quad D_1=(k+p_1)^2-m^2,\quad D_2=(k-p_2)^2-m^2$$

The Basic Tools For The Computation

- `$Apart`: Decompose the propagators to independent ones

$$D_0 = k^2, \quad D_1 = (k+p_1)^2 - m^2, \quad D_2 = (k-p_2)^2 - m^2$$

Taking k^2, $k \cdot p_1$ and $k \cdot p_2$ as the basis of the vector space

$$\texttt{\$Apart}\left[\frac{k\cdot p_1 k\cdot p_2}{D_0\ D_1\ D_2}, \{k^2, k\cdot p_1, k\cdot p_2\}\right]$$

$$= \frac{1}{4}\left[\frac{1}{D_0} - \frac{1}{D_1} + \left(m^2 - p_1^2\right)\frac{1}{D_0\ D_1} + \left(m^2 - p_2^2\right)\frac{1}{D_0\ D_2}\right.$$

$$\left. -2\frac{k\cdot p_1}{D_1\ D_2} + \left(p_2^2 - m^2\right)\frac{1}{D_1\ D_2} + \left(m^2 - p_1^2\right)\left(m^2 - p_2^2\right)\frac{1}{D_0\ D_1\ D_2}\right]$$

The vectors in each term are linear independent, and at most 3 vectors in each term.

The Basic Tools For The Computation

- FIRE: Performing the reduction of Feynman integrals to master integrals JHEP **0810**, 107 (2008) [arXiv:0807.3243 [hep-ph]]

$$F(a_1, \cdots, a_n) = \int \cdots \int \frac{d^d k_1 \cdots d^d k_h}{E_1^{a_1} \cdots E_n^{a_n}}$$

Here k_i, $i = 1, \cdots, h$, are loop momenta and the denominators E_r are either quadratic or linear with respect to the loop momenta k_i of the graph. Irreducible polynomials in the numerator can be represented as denominators raised to negative powers.

- Integration By Parts (IBP) & Master Integrals (MI)

$$\int \cdots \int d^d k_1 d^d k_2 \cdots \frac{\partial}{\partial k_i} \left[\frac{p_j}{E_1^{a_1} \cdots E_n^{a_n}} \right] = 0$$

$$\sum \alpha_i F(a_1 + b_{i,1}, \cdots, a_n + b_{i,n}) = 0$$

$e^+e^- \to J/\psi + \eta_c$

In collaboration with Hai-Rong Dong & Yu Jia

PRD 85, 114018 (2012)

- Large discrepancy between experimental data and Leading-Order NRQCD predictions

  ➜ Experiment

  Belle (2004) : $\sigma[e^+e^- \to J/\psi + \eta_c] \times B_{>2} = 25.6 \pm 2.8 \pm 3.4$ fb.

  BABAR (2005) : $\sigma[e^+e^- \to J/\psi + \eta_c] \times B_{>2} = 17.6 \pm 2.8^{+1.5}_{-2.1}$ fb.

  ➜ NRQCD at LO in α_s and v

  Braaten, Lee (2003) : $\sigma[e^+e^- \to J/\psi + \eta_c] = 3.7 \pm 1.26$ fb.

  Liu, He, Chao (2003) : $\sigma[e^+e^- \to J/\psi + \eta_c] = 5.5$ fb.

  The two calculations employ different choices of m_c, NRQCD matrix elements, and α_s.

  Braaten and Lee also include QED effects.

$e^+e^- \to J/\psi + \eta_c$

- Some Possible Explanations
  - Some of the $J/\psi + \eta_c$ data sample may consist of $J/\psi + J/\psi$ events.
  - Some of the data sample may be from $e^+e^- \to J/\psi + \mathrm{glueball}$.
- α_s Corrections to $e^+e^- \to J/\psi + \eta_c$
  - An important step in resolving the discrepancy.
    - Zhang, Gao, Chao (2005) found that corrections at NLO in α_s yield a K factor of about 1.96.
    - Confirmed by Gong and Wang (2007).
  - Not enough by itself to bring theory into agreement with experiment.
- Relativistic Corrections to $e^+e^- \to J/\psi + \eta_c$
  - Direct and Indirec Corrections.
  - Corrections at NLO in α_s plus relativistic corrections may bring theory into agreement with experiment.
    - Confirmed by He, Fan, Chao (2007).

$e^+e^- \to J/\psi + \eta_c$

- $\sigma_{\text{total}}(e^+e^- \to J/\psi + \eta_c)$ consists of:

5.4 fb	Leading order in α_s and v^2 (including indir. rel. corr., but without QED contribution)
1.0 fb	QED contribution
2.9 fb	Direct relativistic corrections
6.9 fb	Corrections of NLO in α_s
1.4 fb	Interference between rel. corr. and corr. of NLO in α_s
17.6 fb	Total

- The uncalculated correction to $\sigma(e^+e^- \to J/\psi + \eta_c)$ of relative order $\alpha_s v^2$ is potentially large, as is the uncalculated correction of relative order α_s^4. While the calculation of the former correction may be feasible, the calculation of the latter correction is probably beyond the current state of the art.
  [arXiv:1010.5827v3 [hep-ph] 11 Feb 2011]

$e^+e^- \to J/\psi + \eta_c$

- EM form factor

$$\langle J/\psi(P_1,\lambda) + \eta_c(P_2)|J^\mu_{\rm em}|0\rangle = i\,G(s)\,\epsilon^{\mu\nu\rho\sigma}P_{1\nu}P_{2\rho}\varepsilon^*_\sigma(\lambda)$$

$$\sigma[e^+e^- \to J/\psi + \eta_c] = \frac{4\pi\alpha^2}{3}\left(\frac{|\mathbf{P}|}{\sqrt{s}}\right)^3 |G(s)|^2$$

- NRQCD factorization formula

$$\begin{aligned} G(s) \quad = \quad & \sqrt{4M_{J/\psi}M_{\eta_c}}\langle J/\psi|\psi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}\chi|0\rangle\langle\eta_c|\psi^\dagger\chi|0\rangle \\ & \left[c_0 + c_{2,1}\langle v^2\rangle_{J/\psi} + c_{2,2}\langle v^2\rangle_{\eta_c} + \cdots\right] \end{aligned}$$

$$\langle v^2\rangle_{J/\psi} = \frac{\langle J/\psi(\lambda)|\psi^\dagger(-\frac{i}{2}\overleftrightarrow{\mathbf{D}})^2\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}(\lambda)\chi|0\rangle}{m_c^2\,\langle J/\psi(\lambda)|\psi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}(\lambda)\chi|0\rangle} \qquad \langle v^2\rangle_{\eta_c} = \frac{\langle\eta_c|\psi^\dagger(-\frac{i}{2}\overleftrightarrow{\mathbf{D}})^2\chi|0\rangle}{m_c^2\,\langle\eta_c|\psi^\dagger\chi|0\rangle}$$

$e^+e^- \to J/\psi + \eta_c$

- Total cross section

$$\sigma[e^+e^- \to J/\psi + \eta_c] = \sigma_0 + \sigma_2 + \mathcal{O}(\sigma_0 v^4)$$

$$\sigma_0 = \frac{8\pi\alpha^2 m_c^2 (1-4r)^{3/2}}{3} \langle \mathcal{O}_1 \rangle_{J/\psi} \langle \mathcal{O}_1 \rangle_{\eta_c} |c_0|^2,$$

$$\sigma_2 = \frac{4\pi\alpha^2 m_c^2 (1-4r)^{3/2}}{3} \langle \mathcal{O}_1 \rangle_{J/\psi} \langle \mathcal{O}_1 \rangle_{\eta_c}$$

$$\left\{ \left(\frac{1-10r}{1-4r} |c_0|^2 + 4\,\mathrm{Re}[c_0 c_{2,1}^*] \right) \langle v^2 \rangle_{J/\psi} + \left(\frac{1-10r}{1-4r} |c_0|^2 + 4\,\mathrm{Re}[c_0 c_{2,2}^*] \right) \langle v^2 \rangle_{\eta_c} \right\}$$

- Leading-order coefficients

$$c_0^{(0)} = \frac{32\pi C_F e_c \alpha_s}{N_c\, m_c s^2}, \quad c_{2,1}^{(0)} = c_0^{(0)} \left[\frac{3-10r}{6} + \left(1 - \frac{16}{9} r\right) \epsilon + \mathcal{O}\left(\epsilon^2\right) \right]$$

$$c_{2,2}^{(0)} = c_0^{(0)} \left[\frac{2-5r}{3} + \left(\frac{10}{9} - \frac{16}{9} r \right) \epsilon + \mathcal{O}\left(\epsilon^2\right) \right], \qquad r \equiv \frac{4m_c^2}{s}$$

$e^+e^- \to J/\psi + \eta_c$

- Next-to-leading order coefficients

$$c_0^{(1)}\left(r, \frac{\mu_r^2}{s}\right) \underset{\text{asym}}{=} c_0^{(0)} \times \left\{\beta_0\left(-\frac{1}{4}\ln\frac{s}{4\mu_r^2} + \frac{5}{12}\right) + \left(\frac{13}{24}\ln^2 r + \frac{5}{4}\ln 2\ln r - \frac{41}{24}\ln r\right.\right.$$
$$\left.\left. -\frac{53}{24}\ln^2 2 + \frac{65}{8}\ln 2 - \frac{1}{36}\pi^2 - \frac{19}{4}\right) + i\pi\left(\frac{1}{4}\beta_0 + \frac{13}{12}\ln r + \frac{5}{4}\ln 2 - \frac{41}{24}\right)\right\},$$

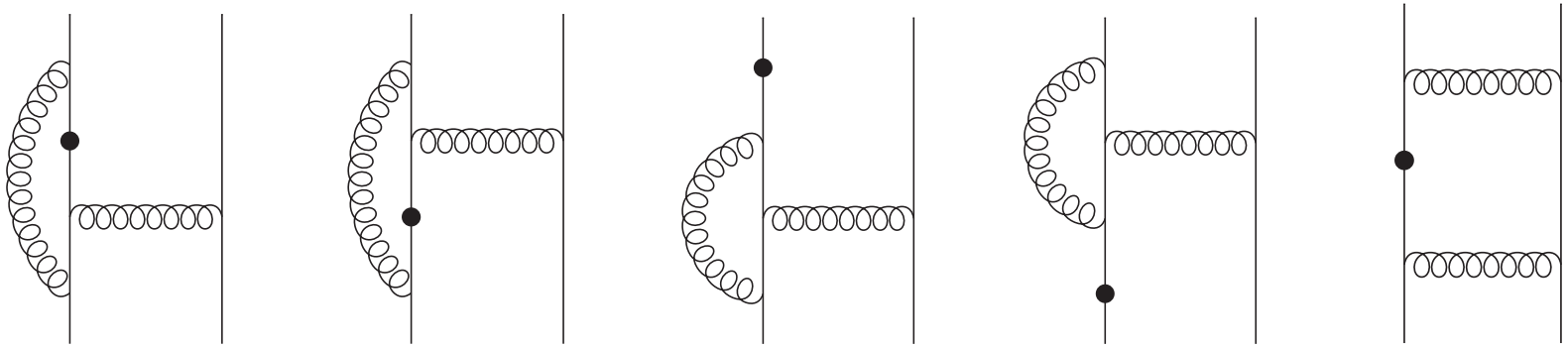
$$c_{2,1}^{(1)}\left(r, \frac{\mu_r^2}{s}, \frac{\mu_f^2}{m_c^2}\right) \underset{\text{asym}}{=} \frac{1}{2}c_0^{(0)} \times \left\{\frac{16}{9}\ln\frac{\mu_f^2}{m_c^2} + \beta_0\left(-\frac{1}{4}\ln\frac{s}{4\mu_r^2} + \frac{11}{12}\right) + \left(\frac{3}{8}\ln^2 r + \frac{19}{12}\ln 2\ln r\right.\right.$$
$$\left.\left. +\frac{31}{24}\ln r - \frac{1}{24}\ln^2 2 + \frac{893}{216}\ln 2 - \frac{5}{36}\pi^2 - \frac{497}{72}\right) + i\pi\left(\frac{1}{4}\beta_0 + \frac{3}{4}\ln r + \frac{19}{12}\ln 2 + \frac{9}{8}\right)\right\},$$

$$c_{2,2}^{(1)}\left(r, \frac{\mu_r^2}{s}, \frac{\mu_f^2}{m_c^2}\right) \underset{\text{asym}}{=} \frac{2}{3}c_0^{(0)} \times \left\{\frac{4}{3}\ln\frac{\mu_f^2}{m_c^2} + \beta_0\left(-\frac{1}{4}\ln\frac{s}{4\mu_r^2} + \frac{2}{3}\right) + \left(\frac{1}{12}\ln^2 r + \frac{11}{12}\ln 2\ln r\right.\right.$$
$$\left.\left. -\frac{1}{24}\ln r - \frac{11}{8}\ln^2 2 + \frac{241}{144}\ln 2 - \frac{1}{8}\pi^2 - \frac{99}{16}\right) + i\pi\left(\frac{1}{4}\beta_0 + \frac{1}{6}\ln r + \frac{11}{12}\ln 2 - \frac{1}{24}\right)\right\}.$$

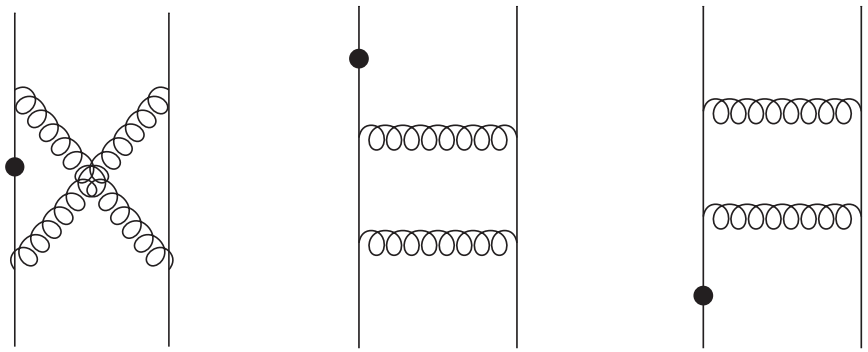
$e^+e^- \to J/\psi + \eta_c$

- Double logarithms for each Feynman diagram

$$(c_0, c_1, c_2) \to -\frac{e r^2 g_s^4}{36 N_c m_c^3 \pi^2}\left[c_0 + c_1 \frac{\boldsymbol{q}_1^2}{m_c^2} + c_2 \frac{\boldsymbol{q}_2^2}{m_c^2}\right] \ln^2 r$$



$(1, 1, \frac{11}{6})$ $(1, 1, \frac{1}{2})$ $(1, 1, \frac{11}{6})$ $(1, 1, \frac{1}{2})$ $(4, 4, -\frac{2}{3})$



$(-3, -\frac{8}{3}, -\frac{17}{6})$ $(16, \frac{32}{3}, \frac{40}{3})$ $(8, 8, 4)$ $(-1, -1, -\frac{11}{6})$ $(-2, -1, -1)$

$e^+e^- \to J/\psi + \eta_c$

- Phenomenology

➜ Numeric parameters

$$\sqrt{s} = 10.58 \text{ GeV}, \quad \alpha(\sqrt{s}) = 1/130.9,$$
$$\langle \mathcal{O}_1 \rangle_{J/\psi} \approx \langle \mathcal{O}_1 \rangle_{\eta_c} = 0.387 \text{ GeV}^3,$$
$$\langle v^2 \rangle_{J/\psi} = 0.223, \quad \langle v^2 \rangle_{\eta_c} = 0.133, \quad \mu_f = m_c$$

➜ Contributions from different parts

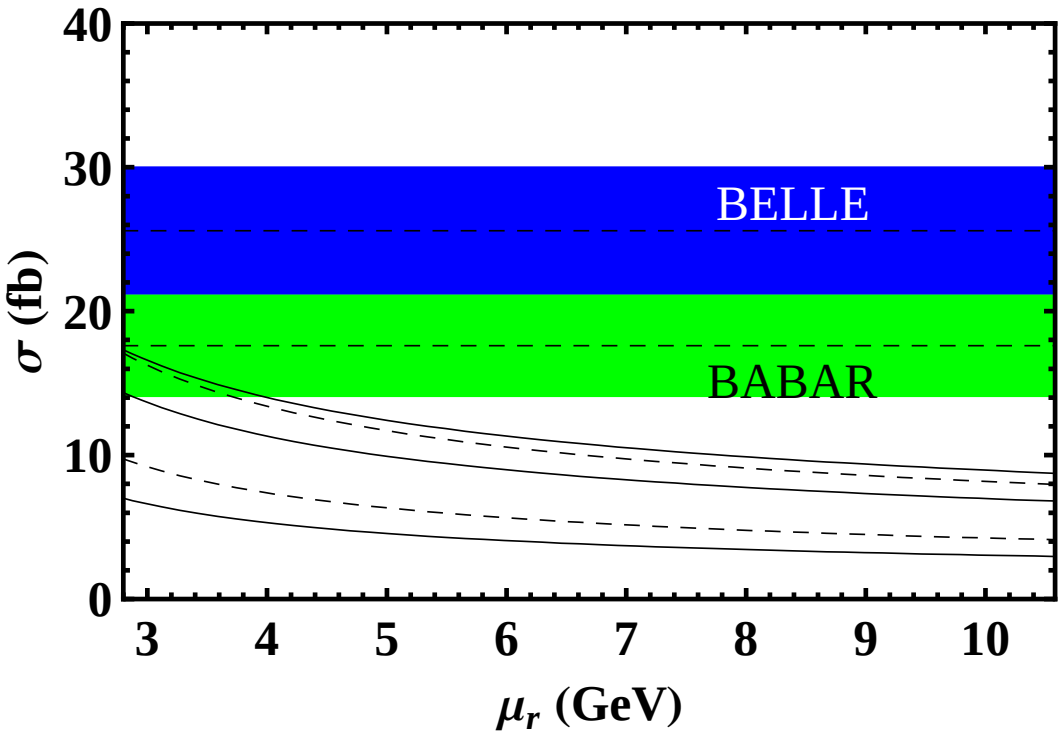
	$\sigma_0^{(0)}$	$\sigma_0^{(1)}$	$\sigma_2^{(0)}$	$\sigma_2^{(1)}$
$\alpha_s(\frac{\sqrt{s}}{2}) = 0.211$	4.40	5.22	1.72	0.73
$\alpha_s(2m_c) = 0.267$	7.00	7.34	2.73	0.24

Individual contributions to the predicted $\sigma[e^+e^- \to J/\psi + \eta_c]$ at $\sqrt{s} = 10.58$ GeV, labeled by powers of α_s and v.

The cross sections are in units of fb.

$e^+e^- \to J/\psi + \eta_c$

➜ Compared with experiments at B factories



The μ-dependence of the cross section for $e^+e^- \to J/\psi + \eta_c$.

The 5 curves from bottom to up are $\sigma_0^{(0)}$ (solid), $\sigma_0^{(0)} + \sigma_2^{(0)}$ (dashed), $\sigma_0^{(0)} + \sigma_0^{(1)}$ (solid), $\sigma_0^{(0)} + \sigma_2^{(0)} + \sigma_0^{(1)}$ (dashed), and $\sigma_0^{(0)} + \sigma_2^{(0)} + \sigma_0^{(1)} + \sigma_2^{(1)}$ (solid) respectively.

$J/\psi \to 3\gamma$

In collaboration with Yu Jia & Wen-Long Sang

arXiv:1210.6337

- The process has not been ovserved for a long time, due to the relative small branching ratio
  - ➜ CLEO-c (2008) : Br[$J/\psi \to 3\gamma$] = $1.2 \pm 0.3 \pm 0.2 \times 10^{-5}$
  - ➜ BESIII (2012) : Br[$J/\psi \to 3\gamma$] = $11.3 \pm 1.8 \pm 2.0 \times 10^{-6}$
- Theoretical prediction and measurement is consistent with each other in high accuracy for ortho-positronium annihilation decay into three photons (hep-ph/0506213)
  - ➜ $\Gamma(\text{theory}) = 7.039979(11)\mu\text{s}^{-1}$ (higher order logarithmic corrections)
  - ➜ $\Gamma(\text{Tokyo}) = 7.0396(12\ \text{stat.})(11\ \text{syst.})\mu\text{s}^{-1}$
  - ➜ $\Gamma(\text{Michigan}) = 7.0404(10\ \text{stat.})(8\ \text{syst.})\mu\text{s}^{-1}$
- NOT happened for $J/\psi \to 3\gamma$
- First, we reproduced the results in Phys. Rev. Lett. **76**, 4903 (1996)

$J/\psi \to 3\gamma$

- Symmetries of the decay tensor

$$M = \epsilon^*_{1\mu_1}\epsilon^*_{2\mu_2}\epsilon^*_{3\mu_3}\epsilon_\alpha M^{\mu_1\mu_2\mu_3\alpha}(k_1, k_2, k_3)$$

The decay tensor $M^{\mu_1\mu_2\mu_3\alpha}$ is a linear combination of terms like $k_a^{\mu_1}k_b^{\mu_2}k_c^{\mu_3}k_d^{\alpha}$, $k_a^{\mu_1}k_b^{\mu_2}g^{\mu_3\alpha}$ and $g^{\mu_1\mu_2}g^{\mu_3\alpha}$

	$k_a^{\mu_1}k_b^{\mu_2}k_c^{\mu_3}k_d^{\alpha}$	$k_a^{\mu_1}k_b^{\mu_2}g^{\mu_3\alpha}$	$g^{\mu_1\mu_2}g^{\mu_3\alpha}$
Generally	81	54	3
$\epsilon_{a\mu}k_a^{\mu} = 0$	21	30	3
Permutation	4	6	1

$$M^{\mu_1\mu_2\mu_3\alpha}(k_1, k_2, k_3) = \sum_{S_3}\mathcal{M}^{\mu_1\mu_2\mu_3\alpha}(k_1, k_2, k_3)$$

$J/\psi \to 3\gamma$

- Symmetries of the decay tensor

Gauge invariance requires

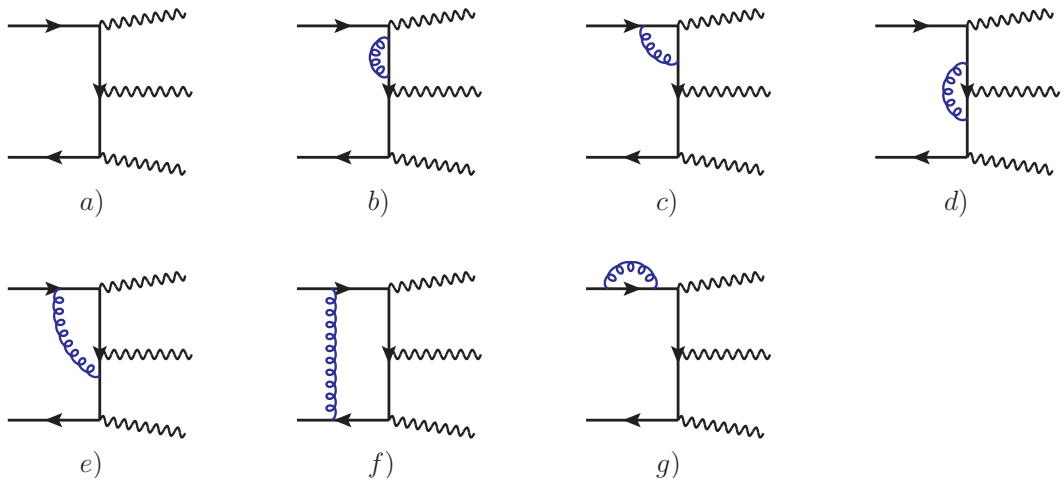
$$k_{1\mu_1} M^{\mu_1\mu_2\mu_3\alpha}(k_1, k_2, k_3) = 0$$

Finally, we left with 3 independent reduce amplitudes A_1, A_2 and A_3

$$\begin{aligned}
& \mathcal{M}^{\mu_1\mu_2\mu_3\alpha}(k_1, k_2, k_3) \\
= \quad & A_1(k_1, k_2, k_3) \frac{1}{k_1 \cdot k_3} \Big(\frac{k_3^{\mu_1} k_1^{\mu_3}}{k_1 \cdot k_3} - g^{\mu_1\mu_3} \Big) k_1^{\alpha} \Big(\frac{k_3^{\mu_2}}{k_2 \cdot k_3} - \frac{k_1^{\mu_2}}{k_1 \cdot k_2} \Big) \\
& + A_2(k_1, k_2, k_3) \Big\{ \frac{1}{k_2 \cdot k_3} \Big(\frac{k_1^{\alpha} k_3^{\mu_1}}{k_1 \cdot k_3} - g^{\alpha\mu_1} \Big) \Big(\frac{k_1^{\mu_2} k_2^{\mu_3}}{k_1 \cdot k_2} - g^{\mu_2\mu_3} \Big) \\
& \qquad + \frac{1}{k_1 \cdot k_3} \Big(\frac{k_1^{\mu_2}}{k_1 \cdot k_2} - \frac{k_3^{\mu_2}}{k_2 \cdot k_3} \Big) (k_1^{\mu_3} g^{\alpha\mu_1} - k_1^{\alpha} g^{\mu_1\mu_3}) \Big\} \\
& + A_3(k_1, k_2, k_3) \frac{1}{k_1 \cdot k_3} \Big(\frac{k_1^{\alpha} k_3^{\mu_1}}{k_1 \cdot k_3} - g^{\alpha\mu_1} \Big) \Big(\frac{k_3^{\mu_2} k_2^{\mu_3}}{k_2 \cdot k_3} - g^{\mu_2\mu_3} \Big)
\end{aligned}$$

$J/\psi \to 3\gamma$

- Feynman diagrams



- Reduced amplitudes at tree level

$$A_1 = -\frac{8\sqrt{2}e^3 e_c^3 \boldsymbol{q}^2}{(D-1)m_c}\frac{\bar{x}_1\bar{x}_2^2\bar{x}_3}{x_1^2 x_2 x_3}$$

$$A_2 = -\frac{2\sqrt{2}e^3 e_c^3}{(D-1)m_c}\frac{\bar{x}_1\bar{x}_2\bar{x}_3}{x_1^2 x_2^2 x_3^2}\left[\begin{array}{l} -2(D-1)m_c^2 x_1 x_2 x_3 \\ -\boldsymbol{q}^2\left(x_1\left(x_2\left(Dx_3-8\right)-4x3\right)-4x_2^2 x_3+8\right)\end{array}\right]$$

$$A_3 = -\frac{8\sqrt{2}e^3 e_c^3 \boldsymbol{q}^2}{(D-1)m_c}\frac{\bar{x}_1\bar{x}_2^2}{x_1 x_2^2 x_3}$$

$J/\psi \to 3\gamma$

- NRQCD factorization at the amplitude level

$$A_i = \sqrt{2M}\langle 0|\chi^\dagger \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*\psi|J/\psi\rangle\left(c_{i0} + c_{i2}\langle v^2\rangle + \cdots\right)$$

- Decay rate

$$\begin{aligned}\Gamma &= \frac{1}{3!}\frac{1}{2M}\int \frac{d^3k_1}{(2\pi)^3 2\omega_1}\frac{d^3k_2}{(2\pi)^3 2\omega_2}\frac{d^3k_3}{(2\pi)^3 2\omega_3}(2\pi)^4\delta(P-k_1-k_2-k_3)|M|^2 \\ &= \frac{M}{1536\pi^3}\int_0^1 dx_1\int_{1-x_1}^1 dx_2\,|M|^2 \qquad \text{with } x_i \equiv \frac{2\omega_i}{M}\end{aligned}$$

- Numerical results

$$\begin{aligned}\Gamma(J/\psi\to 3\gamma) &= \frac{|\langle 0|\chi^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*\psi|J/\psi\rangle|^2}{m_c^2}\frac{8(\pi^2-9)e_c^6\alpha^3}{9}\Bigg\{1 - 12.630\,\frac{\alpha_s}{\pi} \\ &\quad + \left[\frac{132-19\pi^2}{12(\pi^2-9)} + \left(\frac{8}{9}\ln\frac{\mu_f^2}{m_c^2} + 68.913\right)\frac{\alpha_s}{\pi}\right]\langle v^2\rangle_{J/\psi}\Bigg\}\end{aligned}$$

$J/\psi \to 3\gamma$

- Numerical parameters

$$\alpha = 1/137 \qquad m_c = 1.4\ \mathrm{GeV} \qquad \mu_f = m_c$$
$$\left|\langle 0|\chi^\dagger \boldsymbol{\sigma}\cdot\boldsymbol{\epsilon}^*\psi|J/\psi\rangle\right|^2 = 0.446\ \mathrm{GeV}^3 \quad \langle v^2\rangle = 0.223$$

- Contributions from different parts

	$\mathrm{Br}_0^{(0)}$	$\mathrm{Br}_0^{(1)}$	$\mathrm{Br}_2^{(0)}$	$\mathrm{Br}_2^{(1)}$	$\mathrm{Br}_{\mathrm{tot}}$
$\alpha_s(m_c) = 0.388$	6.46	-7.67	-10.08	12.26	0.979

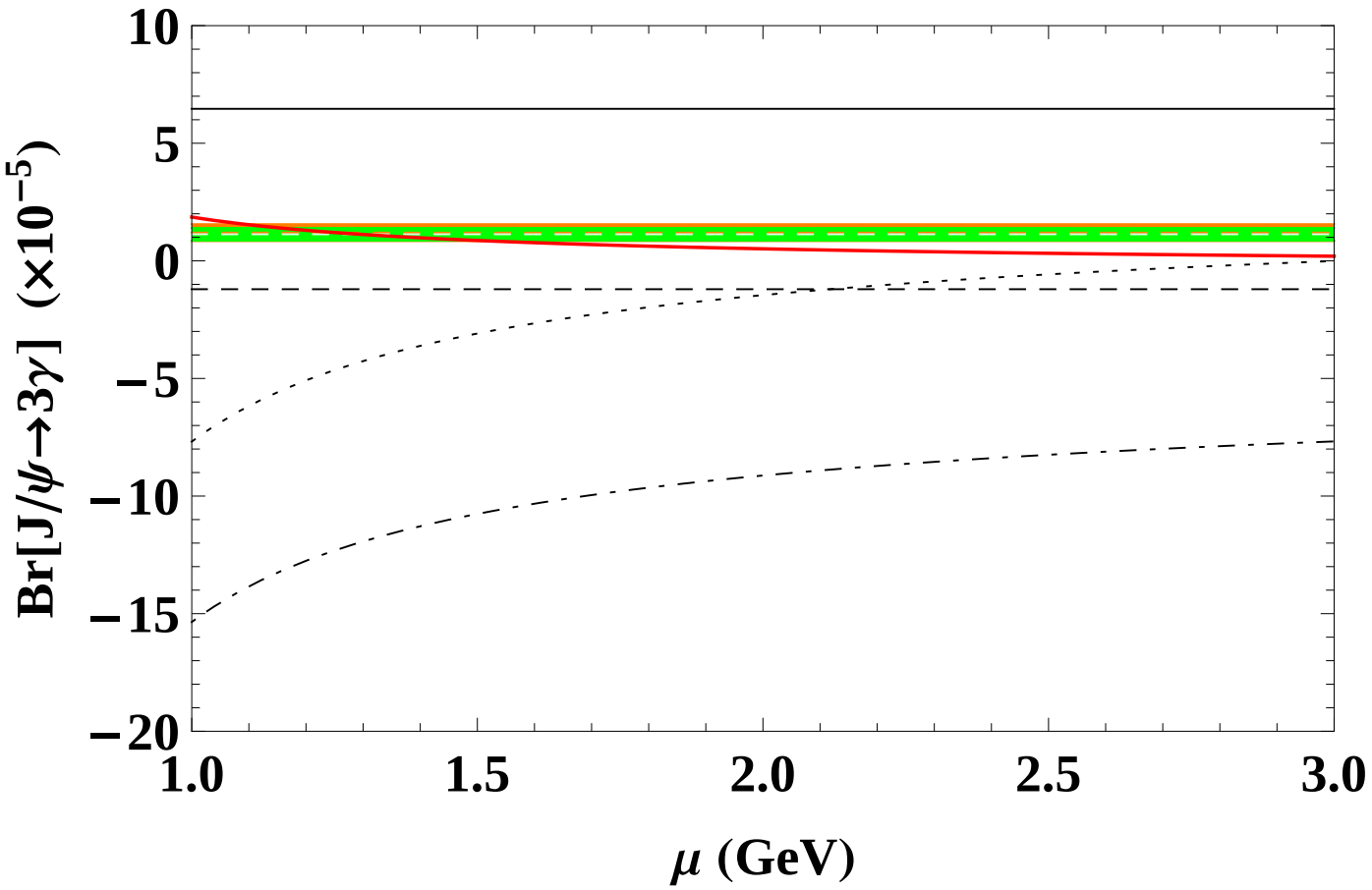
Individual contributions to the predicted $\mathrm{Br}[J/\psi \to 3\gamma]$, labeled by powers of α_s and v. The branching ration are in units of 10^{-5}.

➜ CLEO-c (2008) : Br[$J/\psi \to 3\gamma$] = $1.2 \pm 0.3 \pm 0.2 \times 10^{-5}$

➜ BESIII (2012) : Br[$J/\psi \to 3\gamma$] = $11.3 \pm 1.8 \pm 2.0 \times 10^{-6}$

$J/\psi \to 3\gamma$

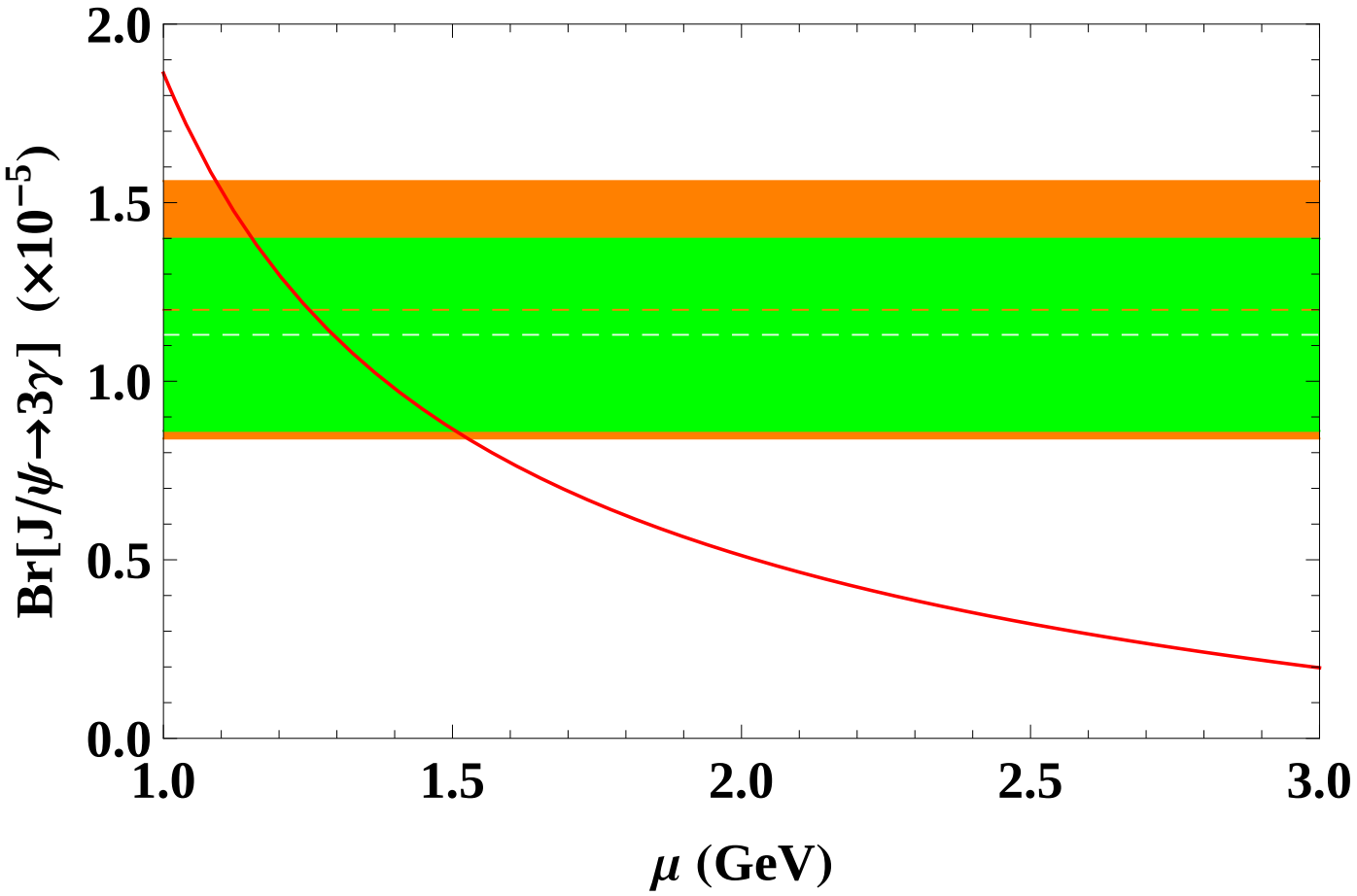
- Theoretical prediction v.s. Experimental data



The μ-dependence of the branching ratio for $J/\psi \to 3\gamma$

$J/\psi \to 3\gamma$

- Theoretical prediction v.s. Experimental data



The μ-dependence of the branching ratio for $J/\psi \to 3\gamma$

Summary and Conclusions

- The method of region expansion can greatly simplify the computation in quarkonium decay and production within the NRQCD framework.
- The $\mathcal{O}(\alpha_s v^2)$ contribution to $e^+e^- \to J/\psi + \eta_c$ modestly enhances the existing NRQCD predictions at the B factory energy.
- The $\mathcal{O}(\alpha_s v^2)$ contribution to $J/\psi \to 3\gamma$ greatly reduce the discrepancy between NRQCD predictions and the experimental data.
- The method can also be easily applied to other quarkonium decay and production processes.

Thanks For Your Attention!